Math 1830 Project Number 2- Continuity of a bizarre function!

Directions: This project is due at the beginning of class Monday September 25. You may discuss it with each other or the professor. You may if you wish work in pairs. Do not wait until the last minute to work on this project!!

Introduction: This project is designed so you will understand why "not picking up one's pencil" is not a sufficient definition of continuity. Recall that a real number x is called a *rational number* if there are integers m and n such that $x = \frac{m}{n}$. We will assume our rational numbers are in lowest terms, i.e. m and n have no common factors. You can assume the following important property of the real numbers:

Fact: Every interval of the real numbers contains both rational and irrational numbers. In particular, between any two different rational numbers there are infinitely many irrational numbers and between any two irrational numbers there are infinitely many rational numbers.

Warmup problems:

1. Write all the rational numbers in the interval $(0, 1)$ which, when written in lowest terms, have denominator equal to 10. (Hint: There should be 4 of them).

2. Do the same for denominator of 19.

3. Consider the irrational number $a = \frac{\sqrt{2}}{2} = .7071067....$

a. Find a $\delta > 0$ so that the interval $(a - \delta, a + \delta)$ contains no fraction with denominator 19 in lowest terms. Repeat for denominator of 31. Be sure to explain how you know your δ works.

b. Find an $\delta > 0$ so that the interval $(a-\delta, a+\delta)$ contains no fractions with denominator $1, 2, 3, 4, 5, \ldots, 19$ in lowest terms. Be sure you explain why your δ works.

c. Suppose a is any irrational number and $N > 0$. Explain why you can always find an $\delta > 0$ so that $(a - \delta, a + \delta)$ contains *no* fractions $\frac{m}{n}$ in lowest terms where $n < N$.

Main problems:

Define a function f on the domain $(0, 1)$ by:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is an irrational number,} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ in lowest terms.} \end{cases}$$

1. Find $f(9/18), f(\sqrt{2}/2), f(.314)$

2. Carefully prove that $f(x)$ is continuous at a if a is irrational and discontinuous at a if a is rational. Use your work in the warmup problems to help you.

3. Can you define $f(0)$ so f will be continuous from the right at 0?