Math 8300- Fall 2006 Final Exam Solutions.

Part I

1. For any $g \in G$, the set gHg^{-1} is a subgroup of order $|H|$, and thus is equal to H by the assumption. Hence $H \trianglelefteq G$.

2a. The conjugacy classes of S_3 are $\{e\}$, $\{(1,2),(2,3),(1,3)\}$, $\{(1,2,3),(1,3,2)\}$ and the conjugacy classes of D_8 are $\{e\}$, $\{r^2\}$, $\{s, sr^2\}$, $\{sr, sr^3\}$, $\{r, r^3\}$.

b. Recall the Sylow subgroups of S_n are iterated wreath products. For S_9, $p = 3$ we can use as generators: $\{(1,2,3),(4,5,6),(7,8,9),(1,4,7)(2,5,8)(3,6,9)\}$.

3. Let G be simple of order 168. By Sylow's theorem the number of Sylow 7 subgroups must be 1 or 8, but G is simple so there are 8. Each has order 7 and contains e plus six elements of order 7. Their pairwise intersections are all trivial and every element of order 7 lies in one so there are $8 \cdot 6 = 48$ elements of order 7.

4. a. $\mathbb{Z}/6\mathbb{Z}$. b. $\mathbb{Z}$ c. $\mathbb{Z}[2i]$ d. $(2, x)$

5. A group G is solvable if there exists $\{e\} \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_k = G$ with G_i/G_{i-1} abelian for all i.

6. To construct a semidirect product, suppose we have two groups H and K and a homomorphism $\phi : H \to \mathrm{Aut}(K)$. Then the semidirect product has as its elements the Cartesian product $K \times H$ with operation given by:

$$(k_1, h_1)(k_2, h_2) := (k_1\phi(h_1)(k_2), h_1h_2).$$

Part II

1. Let $I = \cup_{i=1}^{\infty} I_i$ which is an ideal, thus finitely generated. Say $I =< r_1, r_2, \cdots, r_t >$. Each r_i, being in their union, is in some I_{k_i}. Choose u to be the maximal k_i where $1 \leq i \leq t$. Then every $r_i \in I_u$, so $I = I_u$ and the chain terminates.

2. Suppose not, then $\exists i \in I, j \in J$ with $i \notin P$, $j \notin P$. But $ij \in IJ \subseteq P$ which contradicts P being a prime ideal.

3. In any integral domain all maximal ideals are prime since the quotient by a maximal ideal is a field, so certainly it is an integral domain. Suppose now that (p) is prime but not maximal so $(p) \subset (m) \subset R$. (All ideals are assumed principal.) Now $p \in (m)$ so $p = rm \in (p)$. Thus either r or m is in (p) since it is a prime ideal. We are assuming $m \notin (p)$ so $r = xp$ for some $x \in R$. This gives $p = pxm$ so $xm = 1$ so m is a unit and $(m) = R$. Thus (p) is maximal.

4. If R/I is a field then it has no nontrivial ideals. Thus by the correspondence theorem R has no ideals containing I except I and R, so I is maximal.

5. Let $a, b \in R$. We say d is a g.c.d. of a and b if $d|a, d|b$ and for any e with $e|a, e|b$ we have that $e|d$ as well. Suppose R is a PID so $(a, b) = (d)$ for some d.

Then $a \in (d)$ implies $d|a$ and $b \in (d)$ implies $d|b$ Suppose then that $e|a, e|b$. Then $a = ex, b = ey$ so $a, b \in (e)$ so $d \in (e)$ so $d = er$ for some r and $e|d$. Thus d is a gcd of a and b.

Part III

1. Let P be a Sylow p-subgroup of G. Then PN/N is a Sylow subgroup of G/N (done on HW or with 2nd $\cong$ thm.) But G/N is a p-group so $PN/N = G/N$ so $G = PN$. Since $N \leq Z(G)$ then any commutator $[g_1, g_2] = [p_1n_1, p_2n_2]$ must equal $[p_1, p_2]$ so $G' = [G, G] = [P, P] \leq P$ is a p-group.

2. For $x \in G$ we have $|\mathrm{Class}(x)| = [G : C_G(x)]$ so a two-element conjugacy class has a centralizer of index two which must be normal.

3. First suppose P is a p-group and $N \trianglelefteq P$. N is a union of P-conjugacy classes which each have order a power of p. Since $\{e\}$ is such a class, there are at least $p - 1$ other one-element conjugacy classes in P, i.e. central elements. Thus $p \leq |N \cap Z(P)|$, so the theorem holds for p-groups.

Since G is a finite nilpotent group, it is a direct product of its Sylow subgroups, which are all normal. If P is a Sylow p-subgroup of G then $P \cap N$ is a Sylow p-subgroup of N (also done on homework). So choose $p \mid |N|$ so $\{e\} \neq P \cap N \trianglelefteq P$. Thus $\{e\} \neq (P \cap N) \cap Z(P)$, but $Z(P) \leq Z(G)$ since the terms in a direct product commute with each other so $N \cap Z(G) \neq \{e\}$.

4. Let $|G| = 2006$. Sylow's theorem immediately implies the Sylow 17 and 59 subgroups are normal and, since they are prime order they must be cyclic. Since G also has a subgroup of order 2 which is necessarily a compliment, we have:

$$G \cong (C_{17} \times C_{57}) \rtimes C_2.$$

So we must determine the homomorphisms $\phi : C_2 \to \mathrm{Aut}(C_{17} \times C_{57})$. A cyclic group C_p has a unique automorphisms of order 2, namely the one which inverts every element. Thus there are four possible ϕ, depending on which of C_{17} and C_{57} are inverted by the C_2. If neither are, ϕ is trivial and $G_1 \cong C_{17} \times C_{57} \times C_2 \cong C_{2006}$. Otherwise we get:

$$G_2 := \{g, x, y \mid g^2 = x^{17} = y^{57} = e, gxg = x^{-1}, gyg = y\}.$$

$$G_3 := \{g, x, y \mid g^2 = x^{17} = y^{57} = e, gxg = x, gyg = y^{-1}\}.$$

$$G_4 := \{g, x, y \mid g^2 = x^{17} = y^{57} = e, gxg = x^{-1}, gyg = y^{-1}\}.$$

5. Suppose G is infinite and $H \leq G$ with $[G : H] = n < \infty$. Then G acts on the left cosets of H which gives a nontrivial homomorphism $\phi : G \to S_n$. The kernel of this homomorphism (which is the core of H), is a nontrivial normal subgroup, so G cannot be simple.