Solutions to Linear Algebra Practice Problems 1

1. We can solve $M\vec{x} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$ first.

   Consider the augmented matrix $[M\ b] = \begin{bmatrix} 1 & 3 & -4 & 7 & -1 \\ 2 & 6 & 5 & 1 & -2 \\ 3 & 9 & 4 & 5 & -3 \end{bmatrix}$. Now we perform row operations on the augmented matrix.

   $$\begin{bmatrix} 1 & 3 & -4 & 7 & -1 \\ 2 & 6 & 5 & 1 & -2 \\ 3 & 9 & 4 & 5 & -3 \end{bmatrix} \ r_2 := (-2)r_1 + \widetilde{r_2, r_3} := (-3)r_1 + r_3 \begin{bmatrix} 1 & 3 & -4 & 7 & -1 \\ 0 & 0 & 13 & -13 & 0 \\ 0 & 0 & 16 & -16 & 0 \end{bmatrix}$$

   $$r_2 := \tfrac{1}{13} r_2, r_3 \widetilde{:=} \tfrac{1}{16} r_3, r_3 := r_3 - r_2 \begin{bmatrix} 1 & 3 & -4 & 7 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   $$r_1 \widetilde{:= 4r_2} + r_1 \begin{bmatrix} 1 & 3 & 0 & 3 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   So the solution is

   $$\begin{cases} x_1 + 3x_2 + 3x_4 = -1 \\ x_3 - x_4 = 0 \\ x_2 \text{ and } x_4 \text{ are free.} \end{cases} \tag{1}$$

   So

   $$\begin{cases} x_1 = -1 - 3x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = x_4 \\ x_4 \text{ is free.} \end{cases} \tag{2}$$

   Thus the solution of $Mx = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$ is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 - 3x_2 - 3x_4 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$ where x_2 and x_4 are any numbers.

The solution of $Mx = 0$ is of the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 - 3x_4 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

where x_2 and x_4 are any numbers.

2. (a.)

$$
\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 6 \end{array}\right]
\underset{r_2+\frac{1}{2}r_1}{\sim}
\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ 0 & \frac{3}{2} & -1 & 0 & \frac{1}{2} \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 6 \end{array}\right]
$$

$$
\underset{\frac{2}{3}r_2}{\sim}
\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 6 \end{array}\right]
\underset{r_3+r_2}{\sim}
\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{2}{3} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{4}{3} & -1 & \frac{1}{3} \\ 0 & 0 & -1 & 2 & 6 \end{array}\right]
$$

$$
\underset{\frac{3}{4}r_3}{\sim}
\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{2}{3} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{4} & \frac{1}{4} \\ 0 & 0 & -1 & 2 & 6 \end{array}\right]
\underset{r_4+r_3}{\sim}
\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{2}{3} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{5}{4} & \frac{25}{4} \end{array}\right]
$$

$$
\underset{\frac{5}{4}r_4}{\sim}
\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{2}{3} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]
\underset{r_3+\frac{3}{4}r_4}{\sim}
\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{2}{3} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]
$$

$$
\underset{r_2+\frac{3}{2}r_3}{\sim}
\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]
\underset{r_1+r_2}{\sim}
\left[\begin{array}{cccc|c} 2 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]
$$

$$
\underset{\frac{1}{2}r_1}{\sim}
\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]
$$

Thus $(x_1, x_2, x_3, x_4) = (2, 3, 4, 5)$.

(b) The solution in part (a) implies that

$$
2\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 6 \end{bmatrix}.
$$

3. Consider a linear system whose augmented matrix is of the form

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & 2 & 0 & 1 \\ 3 & 5 & a & b \end{array}\right]$$

(a) For what values of a will the system have a unique solution? What is the solution?(your answer may involve a and b)

(b) For what values of a and b will the system have infinitely many solutions?

(c) For what values of a and b will the system be inconsistent?

Answer:

(a)

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & 2 & 0 & 1 \\ 3 & 5 & a & b \end{array}\right] \quad \overbrace{r_2 + (-1)r_1, r_3 + (-3)r_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & a & b-6 \end{array}\right]$$

$$\overbrace{r_3 + (-2)r_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & a & b-4 \end{array}\right] \quad \overbrace{r_1 + (-1)r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & a & b-4 \end{array}\right]$$

The system will have a unique solution when $a \neq 0$. The solution is $(3, -1, \frac{b-4}{a})$.

(b) The system will have infinitely many solutions if $a = 0$ and $b = 4$.

(c) The system will be inconsistent if $a = 0$ and $b \neq 4$

4. (a)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 6 & 10 \\ 10 & 4 & 10 \\ 10 & 2 & 10 \\ 10 & 8 & 10 \end{bmatrix}.$$

(b) Note that B is a 4×3 matrix and A is a 4×4 matrix. So BA is not defined.

(c) $$-3B = -3 \cdot \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 3 \\ -3 & 3 & -3 \\ -3 & -3 & -3 \\ -3 & -3 & -3 \end{bmatrix}.$$

$$B^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

$$B^T A^T = (AB)^T = \left(\begin{bmatrix} 10 & 6 & 10 \\ 10 & 4 & 10 \\ 10 & 2 & 10 \\ 10 & 8 & 10 \end{bmatrix} \right)^T = \begin{bmatrix} 10 & 10 & 10 & 10 \\ 6 & 4 & 2 & 8 \\ 10 & 10 & 10 & 10 \end{bmatrix}.$$

5. (a) $\begin{bmatrix} -2 & 1 \\ 4 & -2 \\ 0 & 0 \\ -6 & 3 \end{bmatrix} = [v_1 \ v_2]$. We have $v_1 = -2v_2$. So the set of column vectors is linearly dependent.

(b) $\begin{bmatrix} -2 & 1 \\ 4 & -2 \\ 2 & 2 \end{bmatrix}$. The first column vector is not a multiple of the second column vector. So the set of column vectors is linearly independent.

(c)

$$\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix} \ move\ the\ last\ \widetilde{row}\ to\ the\ first\ row \ \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix has four pivot vectors. So the columns of the matrix form a linearly independent set.

(d)

$$\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix} \underbrace{\text{interchange first and third row}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ -4 & -3 & 0 \\ 5 & 4 & 6 \end{bmatrix}$$

$$\underbrace{r_3 + 4r_1, r_4 + (-5)r_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & -3 & 12 \\ 0 & 4 & -9 \end{bmatrix} \underbrace{(-1)r_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & -3 & 12 \\ 0 & 4 & -9 \end{bmatrix}$$

$$\underbrace{r_3 + 3r_2, r_4 + (-4)r_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix} \underbrace{\text{interchange 3rd and 4th row}, \frac{1}{7}r_4} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix has three pivot vectors. So the columns of the matrix form a linearly independent set.

(e)

The column vectors of

$$\begin{bmatrix} -4 & -3 & 1 & 5 & 1 \\ 2 & -1 & 4 & -1 & 2 \\ 1 & 2 & 3 & 6 & -3 \\ 5 & 4 & 6 & -3 & 2 \end{bmatrix}$$

form a dependent set since we have five column vectors in R^4. We will have at least one free variable for the solution of $Ax = 0$.

6. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & -3 \\ -2 & -5 \end{bmatrix}$.

(a) b lies in the span if the linear system associated with the augmented matrix $[A \quad b]$ is consistent. Consider the augmented matrix $[A \quad b] = \begin{bmatrix} 1 & 2 & b_1 \\ -1 & -3 & b_2 \\ -2 & -5 & b_3 \end{bmatrix} \underbrace{r_2 := r_1 + r_2, r_3 := r_3 + 2r_1} \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_1 + b_2 \\ 0 & -1 & 2b_1 + b_3 \end{bmatrix}$

$$\underbrace{r_3 := (-1)r_2 + r_3} \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_1 + b_2 \\ 0 & 0 & b_1 - b_2 + b_3 \end{bmatrix}$$

So this system is consistent if $b_1 - b_2 + b_3 = 0$. Thus $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ lies in the span of the column vectors of A if $b_1 - b_2 + b_3 = 0$.

(b) Now $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. So $b_1 - b_2 + b_3 = 1 - 2 + 3 = 2 \neq 0$. Thus $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is not in the span of the column vectors of A.

7. (a)

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 1 & a+1 & 3 \\ 1 & a & a+1 \end{bmatrix} \underbrace{r_2 + (-1)r_1, r_3 + (-1)r_1}_{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & a & 1 \\ 0 & a-1 & a-1 \end{bmatrix}$$

$$\frac{1}{a-1} r_3 \ \underbrace{if a - 1}_{\sim} \neq 0, r_2 \leftrightarrow r_3 \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & a & 1 \end{bmatrix} \underbrace{r_3 + (-a)r_2}_{\sim} \leftrightarrow r_3 \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1-a \end{bmatrix}$$

Thus the column vectors are independent if $a \neq 1$. The column vectors are dependent if $a = 1$.

8. First, note that

$$\begin{aligned} T(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}) =& T(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = T(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) + T(2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) + T(3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) \\ =& T(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) + 2T(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) + 3T(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = T(e_1) + 2T(e_2) + 3T(e_3). \end{aligned}$$

We need to find $T(e_1)$, $T(e_2)$ and $T(e_3)$.

Since T is linear, we have $T(e_1 + e_2) = T(e_1) + T(e_2)$, $T(e_1 - e_2) = T(e_1) - T(e_2)$ and $T(e_1 + e_2 + e_3) = T(e_1) + T(e_2) + T(e_3)$. The conditions $T(e_1 + e_2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $T(e_1 - e_2) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $T(e_1 + e_2 + e_3) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ can be written as

$$\begin{cases} T(e_1) + T(e_2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ T(e_1) - T(e_2) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ T(e_1) + T(e_2) + T(e_3) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \end{cases} \tag{3}$$

Adding $T(e_1)+T(e_2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $T(e_1)-T(e_2) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, we get $2T(e_1) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $T(e_1) = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$. Similarly, $T(e_1) + T(e_2) - (T(e_1) - T(e_2)) = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$. So $2T(e_2) = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} -\frac{1}{2} \\ -2 \end{bmatrix}$. From $T(e_1) + T(e_2) + T(e_3) - (T(e_1) + T(e_2)) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, we get $T(e_3) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Hence $T(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}) = T(e_1)+2T(e_2)+3T(e_3) = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} + 2\cdot \begin{bmatrix} -\frac{1}{2} \\ -2 \end{bmatrix} + 3\cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -6 \end{bmatrix}$.