Solutions to Linear Algebra Practice Problems 1

1. (a.)

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 6 \end{array}\right] \widetilde{r_2 + \frac{1}{2}r_1} \left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ 0 & \frac{3}{2} & -1 & 0 & \frac{1}{2} \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 6 \end{array}\right]$$

$$\widetilde{\frac{2}{3}r_2} \left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 6 \end{array}\right] \widetilde{r_3 + r_2} \left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{2}{3} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{4}{3} & -1 & \frac{1}{3} \\ 0 & 0 & -1 & 2 & 6 \end{array}\right]$$

$$\widetilde{\frac{3}{4}r_3} \left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{2}{3} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{4} & \frac{1}{4} \\ 0 & 0 & -1 & 2 & 6 \end{array}\right] \widetilde{r_4 + r_3} \left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{2}{3} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{5}{4} & \frac{25}{4} \end{array}\right]$$

$$\widetilde{\frac{5}{4}r_4} \left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{2}{3} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 5 \end{array}\right] \widetilde{r_3 + \frac{3}{4}r_4} \left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{2}{3} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

$$\widetilde{r_2 + \frac{3}{2}r_3} \left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right] \widetilde{r_1 + r_2} \left[\begin{array}{cccc|c} 2 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

$$\widetilde{\frac{1}{2}r_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

Thus $(x_1, x_2, x_3, x_4) = (2, 3, 4, 5)$.

(b) The solution in part (a) implies that

$$2\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 6 \end{bmatrix}.$$

2. $M = \begin{bmatrix} 1 & 3 & -4 & 7 \\ 2 & 6 & 5 & 1 \\ 3 & 9 & 4 & 5 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 3 & -4 & 7 \\ 2 & 6 & 5 & 1 \\ 3 & 9 & 4 & 5 \end{bmatrix} \quad \underset{r_2 - 2r_1,\, r_3 + (-3)r_1}{\sim} \begin{bmatrix} 1 & 3 & -4 & 7 \\ 0 & 0 & 13 & -13 \\ 0 & 0 & 16 & -16 \end{bmatrix}$$

$$\underset{\frac{1}{13}r_2,\, \frac{1}{16}r_3,\, r_3 - r_2}{\sim} \begin{bmatrix} 1 & 3 & -4 & 7 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \underset{r_1 + 4r_2}{\sim} \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the reduced row echelon form of M is is $\begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The solution is $x_1 + 3x_2 + 3x_4 = 0$ and $x_3 - x_4 = 0$. Thus $x_1 = -3x_2 - 3x_4$, $x_3 = x_4$.

Any solution is of the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 - 3x_4 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

where x_2 and x_4 are any numbers.

3. (a)

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & 2 & 0 & 1 \\ 3 & 5 & a & b \end{array}\right] \quad \underset{r_2 + (-1)r_1,\, r_3 + (-3)r_1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & a & b-6 \end{array}\right]$$

$$\underset{r_3 + (-2)r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & a & b-4 \end{array}\right] \quad \underset{r_1 + (-1)r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & a & b-4 \end{array}\right]$$

The system will have a unique solution when $a \neq 0$. The solution is $(3, -1, \frac{b-4}{a})$.

(b) The system will have infinitely many solutions if $a = 0$ and $b = 4$.

(c) The system will be inconsistent if $a = 0$ and $b \neq 4$

4.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 6 & 10 \\ 10 & 4 & 10 \\ 10 & 2 & 10 \\ 10 & 8 & 10 \end{bmatrix}.$$

$$B^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

$$B^T A^T = (AB)^T = \left(\begin{bmatrix} 10 & 6 & 10 \\ 10 & 4 & 10 \\ 10 & 2 & 10 \\ 10 & 8 & 10 \end{bmatrix} \right)^T = \begin{bmatrix} 10 & 10 & 10 & 10 \\ 6 & 4 & 2 & 8 \\ 10 & 10 & 10 & 10 \end{bmatrix}.$$

5.

$$\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix} \widetilde{move\ the\ last\ row\ to\ the\ first\ row} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix has four pivot vectors. So the columns of the matrix form a linearly independent set.

$$\begin{bmatrix}-4 & -3 & 0\\ 0 & -1 & 4\\ 1 & 0 & 3\\ 5 & 4 & 6\end{bmatrix} \underbrace{\text{interchange first and third row}} \begin{bmatrix}1 & 0 & 3\\ 0 & -1 & 4\\ -4 & -3 & 0\\ 5 & 4 & 6\end{bmatrix}$$

$$\underbrace{r_3 + 4r_1, r_4 + (-5)r_1} \begin{bmatrix}1 & 0 & 3\\ 0 & -1 & 4\\ 0 & -3 & 12\\ 0 & 4 & -9\end{bmatrix} \underbrace{(-1)r_2} \begin{bmatrix}1 & 0 & 3\\ 0 & 1 & -4\\ 0 & -3 & 12\\ 0 & 4 & -9\end{bmatrix}$$

$$\underbrace{r_3 + 3r_2, r_4 + (-4)r_2} \begin{bmatrix}1 & 0 & 3\\ 0 & 1 & -4\\ 0 & 0 & 0\\ 0 & 0 & 7\end{bmatrix} \underbrace{\text{interchange 3rd and 4th row}, \frac{1}{7}r_4} \begin{bmatrix}1 & 0 & 3\\ 0 & 1 & -4\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

This matrix has three pivot vectors. So the columns of the matrix form a linearly independent set.

The column vectors of

$$\begin{bmatrix}-4 & -3 & 1 & 5 & 1\\ 2 & -1 & 4 & -1 & 2\\ 1 & 2 & 3 & 6 & -3\\ 5 & 4 & 6 & -3 & 2\end{bmatrix}$$

form a dependent set since we have five column vectors in R^4.

6.

$$\begin{bmatrix}0 & 1 & 0 & t\\ -1 & 0 & t & 0\\ 0 & -t & 0 & 1\\ -t & 0 & -1 & 0\end{bmatrix} \underbrace{\text{interchange 1st and 2nd row}, (-1)r_1} \begin{bmatrix}1 & 0 & -t & 0\\ 0 & 1 & 0 & t\\ 0 & -t & 0 & 1\\ -t & 0 & -1 & 0\end{bmatrix}$$

$$\underbrace{r_4 + t \cdot r_1} \begin{bmatrix}1 & 0 & -t & 0\\ 0 & 1 & 0 & t\\ 0 & -t & 0 & 1\\ 0 & 0 & -1-t^2 & 0\end{bmatrix} \underbrace{r_3 + t \cdot r_2} \begin{bmatrix}1 & 0 & -t & 0\\ 0 & 1 & 0 & t\\ 0 & 0 & 0 & 1+t^2\\ 0 & 0 & -1-t^2 & 0\end{bmatrix}$$

$$\underbrace{\frac{1}{1+t^2}r_3, \frac{-1}{1+t^2}r_4, r_3 \leftrightarrow r_4} \begin{bmatrix}1 & 0 & -t & 0\\ 0 & 1 & 0 & t\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

M has four pivot vectors so M is invertible. Note that we have used the fact that $1+t^2 \neq 0$ in the computation.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \widetilde{r_2 + (-1)r_1, r_4 + (-1)r_1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\widetilde{-(1)r_2, r_3 + (-1)r_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \widetilde{-(1)r_3, r_4 + (-1)r_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

A has four pivot vectors so A is invertible.

$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. B has four pivot vectors so B is invertible.

7. (a) Let $A = \begin{bmatrix} 3 & 7 & -3 \\ 3 & 8 & -1 \\ -6 & -14 & 7 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 3 & 7 & -3 & 1 & 0 & 0 \\ 3 & 8 & -1 & 0 & 1 & 0 \\ -6 & -14 & 7 & 0 & 0 & 1 \end{array}\right] \widetilde{r_2 + (-1)r_1, r_3 + 2r_1} \left[\begin{array}{ccc|ccc} 3 & 7 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right]$$

$$\widetilde{r_2 + (-2)r_3, r_1 + 3r_3} \left[\begin{array}{ccc|ccc} 3 & 7 & 0 & 7 & 0 & 3 \\ 0 & 1 & 0 & -5 & 1 & -2 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right] \widetilde{r_1 + (-7)r_2} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 42 & -7 & 17 \\ 0 & 1 & 0 & -5 & 1 & -2 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right]$$

$$\widetilde{\frac{1}{3}r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 14 & \frac{-7}{3} & \frac{17}{3} \\ 0 & 1 & 0 & -5 & 1 & -2 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} 14 & \frac{-7}{3} & \frac{17}{3} \\ -5 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix}$.

(b)Since A^{-1} exists, the equation $Ax = b$ has a solution $x = A^{-1}b$. So the vector $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ lies in the span of the column vectors of A.

8. (a)

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 1 & a+1 & 3 \\ 1 & a & a+1 \end{bmatrix} \overbrace{r_2 + (-1)r_1, r_3 + (-1)r_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & a & 1 \\ 0 & a-1 & a-1 \end{bmatrix}$$

$$\overbrace{\frac{1}{a-1} r_3 \; if a - 1 \neq 0, r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & a & 1 \end{bmatrix} \overbrace{r_3 + (-a)r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1-a \end{bmatrix}$$

Thus M is invertible if $a \neq 1$.

(b) The column vectors are independent if $a \neq 1$. The column vectors are dependent if $a = 1$.

9. First, note that

$$\begin{aligned} T(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}) &= T(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = T(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) + T(2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) + T(3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) \\ &= T(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) + 2T(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) + 3T(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = T(e_1) + 2T(e_2) + 3T(e_3). \end{aligned}$$

Adding $T(e_1 + e_2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $T(e_1 - e_2) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, we get $T(2e_1) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $T(e_1) = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$. Similarly, $T(e_1 + e_2) - T(e_1 - e_2) = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$. So $2T(e_2) = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} -\frac{1}{2} \\ -2 \end{bmatrix}$. From $T(e_1+e_2+e_3) - T(e_1+e_2) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, we get $T(e_3) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Hence $T(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}) = T(e_1) + 2T(e_2) + 3T(e_3) = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} -\frac{1}{2} \\ -2 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -6 \end{bmatrix}$.