Solution to Linear Algebra (Math 2890) Review Problems II

1. (a) Show that the matrix $A = \begin{bmatrix} I & 0 \\ B & I \end{bmatrix}$ is invertible and find its inverse.

(b) Use previous result to find the inverse of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$.

Solution: (a) Let $C = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$. Then C is the inverse of A if $AC = I$. So we have $\begin{bmatrix} X & Y \\ Z & W \end{bmatrix} \begin{bmatrix} I & 0 \\ B & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$
$\Longleftrightarrow \begin{bmatrix} X + BY & Y \\ Z + BW & W \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \Longleftrightarrow \begin{bmatrix} X + BY & Y \\ Z + BW & W \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$
$\Longleftrightarrow X + BY = I, Y = 0, Z + BW = 0$
and $W = I$
$\Longleftrightarrow Y = 0,\ W = I\ X = I - BY = I - B \cdot 0 = I$
and $Z = -BW = -B \cdot I = -B$.
Hence $A^{-1} = \begin{bmatrix} I & 0 \\ -B & I \end{bmatrix}$.

(b) From part (a), we have $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -3 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix}$.

2. (a) Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 8 \\ 2 & 6 & 23 \end{bmatrix}$. Find an LU factorization for A.

Solution: $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 8 \\ 2 & 6 & 23 \end{bmatrix} \widetilde{-r_1 + r_2, -2r_1 + r_3} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 5 \\ 0 & 6 & 17 \end{bmatrix}$
$\widetilde{-3r_1 + r_3} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$.

Consider the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ \underbrace{2}_{divide\ by\ 1} & \underbrace{6}_{divide\ by\ 2} & \underbrace{2}_{divide\ by\ 2} \end{bmatrix}$. We get

$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$. Therefore $A = LU$ where $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$

(b) Use LU decomposition to find the solution of $Ax = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$.

Solution: We have to solve $Ly = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ and $Ux = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = y$.

First, we solve $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

$\iff y_1 = 2,\ y_1 + y_2 = 1$ and $2y_1 + 3y_2 + y_3 = 3$

$\iff y_1 = 2,\ y_2 = 1 - y_1 = 1 - 2 = -1$ and $y_3 = 3 - 2y_1 - 3y_2 = 3 - 2 \cdot 2 - 3 \cdot (-1) = 3 - 4 + 3 = 2$.

So $y = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

Now we solve $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = y = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

$\iff x_1 + 3x_3 = 2,\ 2x_2 + 5x_3 = -1$ and $2x_3 = 2$

$\iff x_3 = 1,\ x_2 = \frac{-1-5x_3}{2} = -3$ and $x_1 = 2 - 3x_3 = 2 - 3 \cdot 1 = -1$

So $x = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$.

3. Find all values of a and b so that the subspace of $\mathbb{R}^4$ spanned by

$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} b \\ 1 \\ -a \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\}$ is two-dimensional.

Solution: Consider the matrix $A = \begin{bmatrix} 0 & b & -2 \\ 1 & 1 & 2 \\ 0 & a & -1 \\ -1 & 1 & 0 \end{bmatrix}$

$\widetilde{interchange\ first\ row\ and\ second\ row} \begin{bmatrix} 1 & 1 & 2 \\ 0 & b & -2 \\ 0 & a & -1 \\ -1 & 1 & 0 \end{bmatrix}$

$\widetilde{r_1 + r_4} \begin{bmatrix} 1 & 1 & 2 \\ 0 & b & -2 \\ 0 & a & -1 \\ 0 & 2 & 2 \end{bmatrix}$

$\widetilde{interchange\ second\ row\ and\ forth\ row} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & a & -1 \\ 0 & b & -2 \end{bmatrix}$

$\widetilde{divide\ second\ row\ by\ 2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & a & -1 \\ 0 & b & -2 \end{bmatrix} \widetilde{-ar_2 + r_3, -br_2 + r_4} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -1-a \\ 0 & 0 & -2-b \end{bmatrix}.$

Now the first and second vectors are pivot vectors. So $rank(A) = 2$ if $-1 - a = 0$ and $-2 - b = 0$.

So $a = -1$ and $b = -2$

4. Let $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \}$. You can assume that $\mathcal{B}$ is a basis for R^3

(a) Which vector x has the coordinate vector $[x]_B = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

Let $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$. So $x = A[x]_B = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

(b) Find the β-coordinate vector of $y = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$.

Solution. We have to solve $Ax = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & 2 & 3 \end{array}\right] \widetilde{\tfrac{1}{2}r_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 3 \end{array}\right] \widetilde{-r_2 + r_3} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 4 \end{array}\right]$$
$$\widetilde{\tfrac{1}{2}r_3} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \widetilde{-3r_2 + r_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

So $[y]_B = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$.

5. Let

$$M = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 2 & 5 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix}$$

Find bases for $Col(M)$ and $Nul(M)$, and then state the dimensions of these subspaces

Solution: $\begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 2 & 5 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix} \widetilde{-r_1 + r_2, -r_2 + r_3} \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 2 \end{bmatrix} \widetilde{-2r_2 + r_3} \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
$\widetilde{-2r_2 + r_3} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

So the first two vectors are pivot vectors and $\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \}$ is a basis for $Col(A)$ and $dim(Col(A)) = 2$.

The solution to $Mx = 0$ is $x_1 + x_3 - x_4 = 0$ and $x_2 + 2x_3 + x_4 = 0$. So

$$x = \begin{bmatrix} -x_3 + x_4 \\ -2x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$ Hence the basis for $Nul(M)$ is $\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \}$ and $dim(Nul(M)) = 2$.

6. Determine which sets in the following are bases for $\mathbb{R}^2$ or $\mathbb{R}^3$. Justify your answer

   (a) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix}$. Solution: Since $\begin{bmatrix} 2 \\ -4 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, the set $\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix} \}$ is dependent.

   (b) $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$. Yes. This set forms a basis since they are independent and span R^3.

   (c) $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.
   This is not a basis since it doesn't span R^3.

   (d) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. This set forms a basis since they are independent and span R^3

   (e) $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. This is not a basis since it is dependent.

7. Let A be the matrix
$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

   (a) Find the characteristic equation of A.

   (b) Find the eigenvalues and a basis of eigenvectors for A.

Solution.

1. $A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix}$.

So $det(A - \lambda I) = (2-\lambda)^3 + 1 + 1 - (2-\lambda) - (2-\lambda) - (2-\lambda) = (4-4\lambda+\lambda^2)(2-\lambda)+2-6+3\lambda = 8-8\lambda+2\lambda^2-4\lambda+4\lambda^2-\lambda^3-4+3\lambda = -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = -(\lambda-1)^2(\lambda-4)$. So the characteristic equation is $-(\lambda-1)^2(\lambda-4) = 0$.

2. Solving the characteristic equation $-(\lambda-1)^2(\lambda-4) = 0$, we get that the eigenvalues are $\lambda = 1$ and $\lambda = 4$.

3. When $\lambda = 1$, we have

$A - \lambda I = \begin{bmatrix} 2-1 & 1 & 1 \\ 1 & 2-1 & 1 \\ 1 & 1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \widetilde{-r_1 + r_2, -r_1 + r_3} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The solution of $(A - I)x = 0$ is $x_1 + x_2 + x_3 = 0$ and $x_1 = -x_2 - x_3$

So $Null(A - I) = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \}$.

The basis for the eigenspace corresponding to eigenvalue 1 is $\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \}$

4. When $\lambda = 4$, we have

$A - \lambda I = \begin{bmatrix} 2-4 & 1 & 1 \\ 1 & 2-4 & 1 \\ 1 & 1 & 2-4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$\widetilde{interchange\ 1st\ row\ and\ 2nd\ row} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$\widetilde{2r_1 + r_2, -r_1 + r_3} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \widetilde{r_2/3, r_2 + r_3} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\widetilde{2r_2 + r_1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

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The solution of $(A-4I)x=0$ is $x_1-x_3=0$ and $x_2-x_3=0$ So $Null(A-I)=\{\begin{bmatrix}x_3\\x_3\\x_3\end{bmatrix}=x_3\begin{bmatrix}1\\1\\1\end{bmatrix}\}$.

The basis for the eigenspace corresponding to eigenvalue 4 is $\{\begin{bmatrix}1\\1\\1\end{bmatrix}\}$

8. Let A be the matrix

$$A=\begin{bmatrix}-3 & -4\\ -4 & 3\end{bmatrix}.$$

(a) Find the eigenvalues and a basis of eigenvectors for A.

(b) Diagonalize the matrix A if possible.

(c) Find the matrix exponential e^A.

Solution.

1. $A-\lambda I=\begin{bmatrix}-3-\lambda & -4\\ -4 & 3-\lambda\end{bmatrix}$.

So $det(A-\lambda I)=(-3-\lambda)(3-\lambda)-16=\lambda^2-25=(\lambda-5)(\lambda+5)$

So the characteristic equation is $(\lambda-5)(\lambda+5)=0$.

2. Solving the characteristic equation $(\lambda-5)(\lambda+5)=0$, we get that the eigenvalues are $\lambda=5$ and $\lambda=-5$.

3. When $\lambda=5$, we have

$A-\lambda I=\begin{bmatrix}-3-5 & -4\\ -4 & 3-5\end{bmatrix}=\begin{bmatrix}-8 & -4\\ -4 & 2\end{bmatrix}\ \widetilde{-r_1/2+r_2}=\begin{bmatrix}-8 & -4\\ 0 & 0\end{bmatrix}$.

The solution of $(A-5I)x=0$ is $-8x_1-4x_2=0$, i.e. and $x_2=-2x_1$ So $Null(A-I)=\{\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\-2x_1\end{bmatrix}=x_1\begin{bmatrix}1\\-2\end{bmatrix}\}$.

The basis for the eigenspace corresponding to eigenvalue 5 is $\{\begin{bmatrix}1\\-2\end{bmatrix}\}$

4. When $\lambda=-5$, we have

$A-\lambda I=\begin{bmatrix}-3+5 & -4\\ -4 & 3+5\end{bmatrix}=\begin{bmatrix}2 & -4\\ -4 & 8\end{bmatrix}\ \widetilde{2r_1+r_2}=\begin{bmatrix}2 & -4\\ 0 & 0\end{bmatrix}$.

The solution of $(A+5I)x = 0$ is $2x_1 - 4x_2 = 0$, i.e. and $x_1 = 2x_2$

So $Null(A-I) = \{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$.

The basis for the eigenspace corresponding to eigenvalue -5 is $\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$

Let $P = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. Then $P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$.

So $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$

and $e^A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} e^5 & 0 \\ 0 & e^{-5} \end{bmatrix}\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{5}e^5 & -\frac{2}{5}e^5 \\ \frac{2}{5}e^{-5} & \frac{1}{5}e^{-5} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{5}e^5 + \frac{4}{5}e^{-5} & -\frac{2}{5}e^5 + \frac{2}{5}e^{-5} \\ -\frac{2}{5}e^5 + \frac{2}{5}e^{-5} & \frac{2}{5}e^5 + \frac{1}{5}e^{-5} \end{bmatrix}$.

9. Find a good approximation for the vector $\begin{bmatrix} .8 & .6 \\ .2 & .4 \end{bmatrix}^n \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ for n very large (say $n = 100$).

1. $A - \lambda I = \begin{bmatrix} .8-\lambda & .6 \\ .2 & .4-\lambda \end{bmatrix}$.

So $det(A-\lambda I) = (.8-\lambda)(.4-\lambda) - .12 = \lambda^2 - 1.2\lambda + .32 - .12 = \lambda^2 - 1.2\lambda + .2 = (\lambda-1)(\lambda-.2)$

So the characteristic equation is $(\lambda-1)(\lambda-.2) = 0$.

2. Solving the characteristic equation $(\lambda-1)(\lambda-.2) = 0$, we get that the eigenvalues are $\lambda = 1$ and $\lambda = .2$.

3. When $\lambda = 1$, we have

$A - \lambda I = \begin{bmatrix} .8-1 & .6 \\ .2 & .4-1 \end{bmatrix} = \begin{bmatrix} -.2 & .6 \\ .2 & -.6 \end{bmatrix} \widetilde{r_1 + r_2} = \begin{bmatrix} -.2 & .6 \\ 0 & 0 \end{bmatrix}$.

The solution of $(A-I)x = 0$ is $-.2x_1 + .6x_2 = 0$, i.e. and $x_1 = 3x_2$ So $Null(A-I) = \{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}\}$.

The basis for the eigenspace corresponding to eigenvalue 1 is $\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\}$

4. When $\lambda = 0.2$, we have

$A - \lambda I = \begin{bmatrix} .8-.2 & .6 \\ .2 & .4-.2 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .2 & .2 \end{bmatrix} \widetilde{-r_1/6 + r_2} = \begin{bmatrix} .6 & .6 \\ 0 & 0 \end{bmatrix}$.

The solution of $(A - 0.2I)x = 0$ is $.6x_1 + .6x_2 = 0$, i.e. and $x_1 = -x_2$

So $Null(A - 0.2I) = \{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$.

The basis for the eigenspace corresponding to eigenvalue .2 is $\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$

Let $P = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$. Then $P^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$.

So $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .2 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$.

Hence $A^n = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (.2)^n \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (.2)^n \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$.

If n is large then $0.2^n \approx 0$ and $A^n \approx \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix} \approx \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$.

Hence $\begin{bmatrix} .8 & .6 \\ .2 & .4 \end{bmatrix}^n \begin{bmatrix} 2 \\ 2 \end{bmatrix} \approx \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{12}{4} \\ \frac{4}{4} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.