1.2 Congruence of Triangles
September 10, 2012

Last Friday, we used ruler and compass in GeoGenra to construct an equilateral triangle, the perpendicular bisector and midpoint of a segment, a perpendicular to a line through a point P on the line, angle bisector, the tangent line through a point P (outside the circle) and a circle. These constructions are called Euclidean constructions. Euclidean constructions consist of the following rules. The drawing of various shapes using only a compass and ruler. No measurement of lengths or angles is allowed. Euclidean constructions consist of repeated application of five basic constructions using the points, lines and circles that have already been constructed. These are:

- Creating the line through two existing points
- Creating the circle through one point with center another point
- Creating the point which is the intersection of two existing, non-parallel lines
- Creating the one or two points in the intersection of a line and a circle (if they intersect)
- Creating the one or two points in the intersection of two circles (if they intersect).
- No measurement of lengths or angles is allowed.

Last Friday, we constructed the diagonal lines of a "kite".

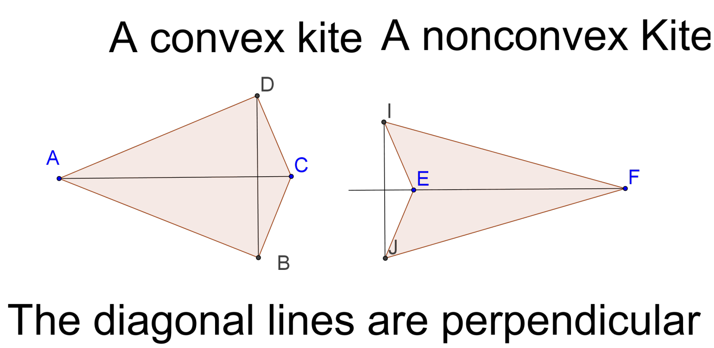

Figure 1:

Definition 1 *A kite is a quadrilateral that has two pairs of congruent adjacent sides.*

From the figure on the left, we can see that A and C are equidistant to B and D. So $\overline{AC}$ is the perpendicular bisector of $\overline{AC}$. Since $\triangle ADB$ is an isosceles triangle, $\overline{AC}$ is also the angle bisector of $\angle A$ and similarly for $\angle C$. We have the following theorem.

Theorem 0.1 *The diagonal of a kite connecting the vertices where the congruent sides intersect bisects the angles at these vertices and is perpendicular bisector of the other diagonal.*

We have uses SAS axiom to deduce the ASA condition. In reality, it is easier to measure the length and is more difficult to measure the angle.The following SSS condition is useful.

Theorem 0.2 ***Side, Side, Side (SSS) Congruent Condition*** *Given a one to one correspondence among the vertices of two triangles, if three sides of one triangle are congruent to the corresponding sides of the second triangle, then the triangles are congruent.*

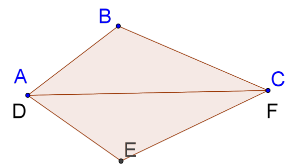

Figure 2:

Proof. Suppose $\triangle ABC$ and $\triangle DEF$ are the given triangles satisfying the given conditions. If we put these two triangles together, we get a kite and $\overline{AC}$ is the angle bisector of $\angle BAE$. This implies that $\angle BAC \cong \angle DEC$. Since $\overline{AB} \cong \overline{DE}$, $\angle BAC \cong \angle DEC$ and $\overline{AC} \cong \overline{DF}$, we have $\triangle ABC \cong \triangle DEF$ by SAS. □

Next we study the congruent condition of right triangles. We first start with the following definition.

Definition 2 *The side opposite the right angle is called the hypotenuse and the other two sides are called legs.*

Since the angles between two legs in a right triangle is 90°, if two legs of two triangles are congruent then they must be congruent. We have the following theorem

Theorem 0.3 ***(The Legs Congruence Condition for right triangle)*** *If two legs of one right triangle are congruent to two legs of another right triangle, then the triangles are congruent.*

Proof. Suppose $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\angle B \cong \angle E = 90°$, $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$. Using SAS, we know that $\triangle ABC \cong \triangle DEF$. □

Next we investigate the relation between the size of an exterior angle of a triangle and the angles inside an triangle.

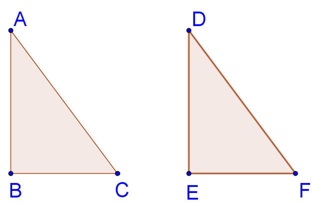

Figure 3:

Definition 3 *When the side of a triangle is extended as in above figure, an angle supplementary to an angle inside the triangle is formed. Such an angle is called the exterior triangle. Note that there are two such congruent exterior angles for each angle. Any angle that is not adjacent to an exterior angle is called a remote interior angle of the exterior angle.*

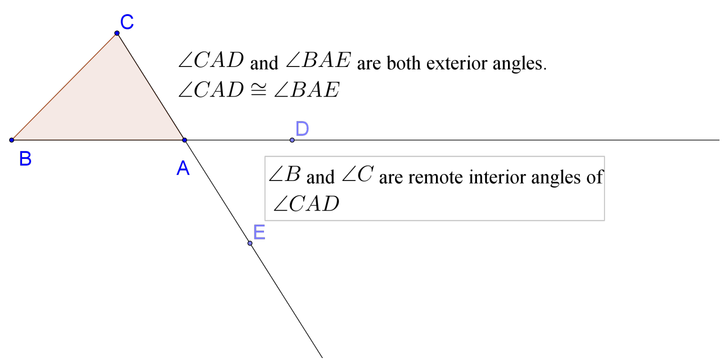

Figure 4:

Now we can prove the following exterior angle theorem

Theorem 0.4 ***The Exterior Angle Theorem*** *An exterior angle of a triangle is greater than either of the remote triangle.*

Proof. Given a ΔABC. We want to show that the exterior angle $m(\angle CAD) > m(\angle C)$ and $m(\angle CAD) > m(\angle B)$. We first pick M to be the midpoint of $\overline{AC}$ and extend $\overline{BM}$ such that $\overline{BM} \cong \overline{MN}$. Since $\overline{CM} \cong \overline{MA}$, $\angle CMB \cong \angle AMN$ and $\overline{BM} \cong \overline{MN}$, we have $\triangle MCB \cong MAN$ by SAS. Hence $\angle MAN \cong \angle BCN$. Obviously, we have $m(\angle MAN) < m(\angle CAD)$. Thus $m(\angle C) \cong m(\angle MAN) < m(\angle CAD)$. Similarly, we can show that $m(\angle CAD) > m(\angle B)$. $\square$

Using the Exterior Angle Theorem, we have the following corollary

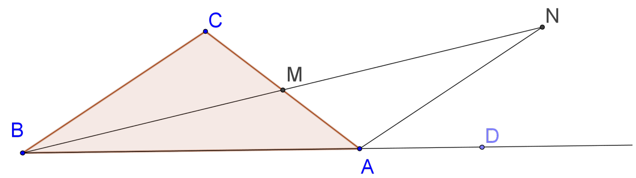

Figure 5:

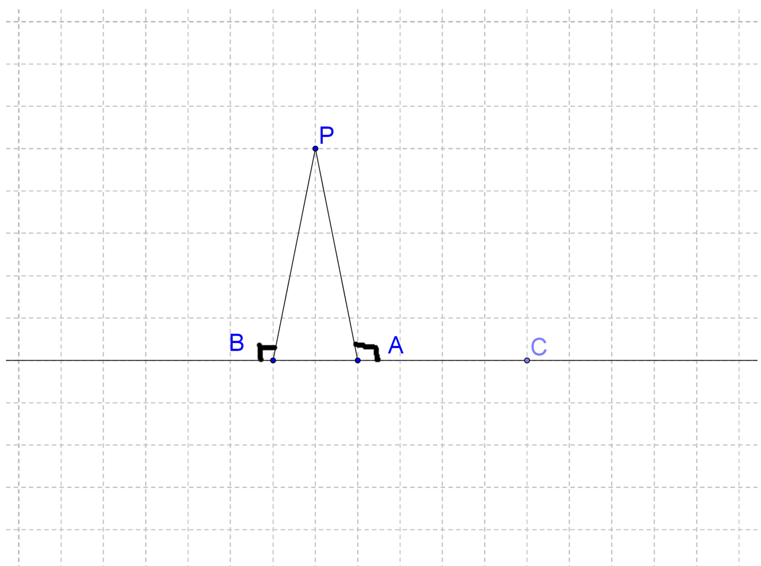

Figure 6:

Corollary 0.1 *Through a point not on a line, there is a unique perpendicular to the line.*

Proof. The existence of such a line is demonstrated in our exercise. We will prove the uniqueness. Suppose there are two such perpendicular lines $\overline{PA}$ and $\overline{PB}$ through a point P. Then $\angle PAC$ is the exterior angle of $\triangle PBA$. By Exterior Angle Theorem, we have $m(\angle PAC) > m(\angle PBA)$. This is impossible since $m(\angle PAC) = m(\angle PBA) = 90°$.

$\square$