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Soliton solutions for the Laplacian co-flow of some G_2-structures with symmetry

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ABSTRACT

We consider the Laplacian "co-flow" of G_2-structures: $\frac{\partial}{\partial t}\psi = -\Delta_d \psi$ where ψ is the dual 4-form of a G_2-structure φ and Δ_d is the Hodge Laplacian on forms. Assuming short-time existence and uniqueness, this flow preserves the condition of the G_2-structure being coclosed ($d\psi = 0$). We study this flow for two explicit examples of coclosed G_2-structures with symmetry. These are given by warped products of an interval or a circle with a compact 6-manifold N which is taken to be either a nearly Kähler manifold or a Calabi–Yau manifold. In both cases, we derive the flow equations and also the equations for soliton solutions. In the Calabi–Yau case, we find all the soliton solutions explicitly. In the nearly Kähler case, we find several special soliton solutions, and reduce the general problem to a single *third order* highly nonlinear ordinary differential equation.

1. Introduction

Flows of G_2-structures were first considered by Robert Bryant in [4]. In particular, Bryant considered the Laplacian flow of G_2-structures: $\frac{\partial}{\partial t}\varphi = \Delta_d \varphi$, where φ is a nondegenerate 3-form defining a G_2-structure, and Δ_d is the Hodge Laplacian on forms. In the case when φ is closed, this condition is preserved under the flow. Using an appropriate choice of inner product on the space of exact 3-forms, one can also show that this flow is the *gradient flow* for the volume functional on the space of torsion-free G_2-structures which was introduced by Hitchin in the arXiv version of [19].

Remark 1.1. Note that since the Hodge Laplacian Δ_d is equal to *minus* the rough Laplacian $\nabla^*\nabla$ plus lower order terms (by the Weitzenböck formula), it can be argued that it is more natural to consider $\frac{\partial}{\partial t}\varphi = -\Delta_d \varphi$ in order for this flow to be qualitatively like a heat equation. However, for *closed* G_2-structures, one can show that $\Delta_d \varphi$ actually only contains first derivatives of φ, so that $\Delta_d \varphi$ and $-\Delta_d \varphi$ are the same, up to lower order terms. Therefore in this case only, both flows are heat-like. The choice $+\Delta_d \varphi$ has the advantage that it is the gradient flow for the Hitchin functional, so it does increase the volume along the flow, and the torsion-free G_2-structures are indeed local maxima of the Hitchin volume functional. The

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fact that $\Delta_d \varphi$ contains only first derivatives of φ when φ is closed can be shown using the general machinery for flows of G_2-structures in [21].

Since this fundamental work by Bryant, the first author has developed several formulas for general flows of G_2-structures in [21]. More recently, there has been work by Xu–Ye [29], Weiss–Witt [28] and Bryant–Xu [5] on the short-time existence and uniqueness of solutions for the Laplacian flow $\frac{\partial}{\partial t}\varphi = \Delta_d \varphi$ for closed G_2-structures.

In this paper we consider the Laplacian "co-flow" of G_2-structures, by which we mean the Laplacian flow of the dual 4-form $\psi = *_\varphi \varphi$. That is, $\frac{\partial}{\partial t}\psi = -\Delta_d \psi$. Since this flow cannot be related to the Hitchin volume functional in any obvious way, and because we do not restrict ourselves to closed G_2-structures, but rather to coclosed G_2-structures, we include a minus sign in front of our Hodge Laplacian to make the equation heat-like. If we assume short-time existence and uniqueness, then this flow preserves the coclosed ($d\psi = 0$) condition, as we discuss in Section 4. The reason we consider this flow is because there exists a general ansatz for a cohomogeneity-one G_2-structure on $M^7 = N^6 \times L^1$ which is a *coclosed* G_2-structure. Here we take the 1-manifold L^1 to be either $\mathbb{R}$ or S^1, and the compact 6-manifold N^6 is taken to be a nearly Kähler 6-manifold or a Calabi–Yau 3-fold.

In Section 2 we review various facts about G_2-structures and their torsion forms. In Section 3 we discuss SU(3)-structures on a 6-manifold N^6, and focus on the special cases of Calabi–Yau and nearly Kähler structures. We also develop some formulas we need later. Section 4 discusses certain properties of the Laplacian co-flow, including its associated soliton solutions. Finally in Sections 5 and 6 we explicitly derive the evolution equations and soliton equations (and discuss their solutions) when N^6 is Calabi–Yau or nearly Kähler, respectively. In particular, we find all the soliton solutions in the Calabi–Yau case. In the nearly Kähler case, we find several special solutions to the soliton equations, including the interesting case of a *sine-cone* metric over a nearly Kähler manifold, which corresponds to a non-torsion-free G_2-structure that is an eigenform of its own Laplacian.

Cohomogeneity-one solitons for the Ricci flow have been extensively studied. Some references (this list is not exhaustive) include [12,13,23].

Note: Throughout this paper, we use $|\cdot|$ and $\langle\cdot,\cdot\rangle$ to denote the pointwise norm and inner product on differential forms and $\|\cdot\|$ and $\langle\langle\cdot,\cdot\rangle\rangle$ to denote the L^2 norm and inner product on forms (the integral of the corresponding pointwise quantity over the manifold).

2. Review of G_2-structures and their torsion

We begin by recalling the definition of a G_2-structure.

Definition 2.1. A 3-form φ on a 7-manifold M^7 is called *nondegenerate* if for any nonzero $X \in T_pM$,

$$0 \neq (X \lrcorner \varphi) \wedge (X \lrcorner \varphi) \wedge \varphi.$$

A smooth nondegenerate 3-form is also called a G_2-*structure*. If φ is a G_2-structure, then there is a unique metric $g = g_\varphi$ and orientation such that if $\mathrm{vol} = \mathrm{vol}_\varphi$ is the volume form associated to that metric and orientation, then for any point $p \in M$ and any vectors $X, Y \in T_pM$, we have

$$-\frac{1}{6}(X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi = g(X, Y)\mathrm{vol}_\varphi.$$

See Bryant [4] for a proof. Note that we are using the opposite orientation of [3,4]. Let $*_\varphi$ be the Hodge star operator of g_φ with the orientation induced by φ. We will often write $*_\varphi$ as $*_7$ to indicate the dimension of the manifold M^7. We will always write ψ to mean the dual 4-form $\psi = *_\varphi \varphi$.

There are various natural conditions on G_2-structures that one can consider.

Definition 2.2. A G_2-structure φ is called *closed* if $d\varphi = 0$, *coclosed* if $d\psi = 0$, and torsion-free if $\Delta_d \varphi = 0$ (or equivalently if $\Delta_d \psi = 0$).

The space of forms on M^7 decomposes into irreducible subspaces under the action of G_2, and this allows us to define the *torsion forms* of a G_2-structure. In particular we have $\Lambda^4 = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27}$ and $\Lambda^5 = \Lambda^5_7 \oplus \Lambda^5_{14}$. Precise descriptions of these subspaces, which we will not require here, can be found in [4,20,21].

Definition 2.3. If φ is a G_2-structure on a 7-manifold, with associated 4-form ψ, then there are unique forms $\tau_0, \tau_1, \tau_2, \tau_3$, called the *torsion forms* of the G_2-structure, where τ_k is a k-form, such that

$$d\varphi = \tau_0 \psi + 3\tau_1 \wedge \varphi + *_\varphi \tau_3,$$
$$d\psi = 4\tau_1 \wedge \psi + *_\varphi \tau_2. \qquad (2.1)$$

We can recover the torsion forms using the following identities:

$$\tau_0 = \frac{1}{7} *_7 (\varphi \wedge d\varphi), \tag{2.2}$$

$$\tau_1 = \frac{1}{12} *_7 (\varphi \wedge *_7 d\varphi) = \frac{1}{12} *_7 (\psi \wedge *_7 d\psi). \tag{2.3}$$

See [4] or [21] for a more detailed discussion about the torsion forms, including the derivation of the above equations. The torsion forms were first considered by Fernàndez–Gray [15] and are also discussed in detail in [7] and [18], for example. When the four torsion forms vanish (equivalently when φ is closed and coclosed) the G_2-structure is called torsion-free and it can be shown that the Riemannian holonomy of the metric g_φ is contained in G_2, and that g_φ is Ricci-flat.

3. SU(3)-structures and their associated G_2-structures

Let N^6 be a smooth 6-manifold. An SU(3)-structure on N^6 is a reduction of the structure group from $GL(6, \mathbb{R})$ to SU(3). Such manifolds come equipped with an almost complex structure J, a Riemannian metric g with respect to which J is orthogonal, and a particular choice of nowhere vanishing smooth complex-valued 3-form Ω of type $(3, 0)$. The metric and the almost complex structure together determine the Kähler form $\omega(X, Y) = g(JX, Y)$, which is a real 2-form of type $(1, 1)$. At each point on N, the magnitude of Ω can be fixed by the requirement that these structures are related by the following equation:

$$\mathrm{vol}_N = \frac{\omega^3}{3!} = \frac{i}{8}\Omega \wedge \bar{\Omega} = \frac{1}{4}\mathrm{Re}(\Omega) \wedge \mathrm{Im}(\Omega). \tag{3.1}$$

Note that if we change Ω to $e^{i\theta}\Omega$, for some phase function $e^{i\theta}$ which can vary on N, then we get the same U(3)-structure but a different SU(3)-structure.

Remark 3.1. For a manifold N^6 equipped with an SU(3)-structure, near each point of N^6 we can find a local unitary coframe of complex-valued 1-forms (ξ_1, ξ_2, ξ_3) for which

$$\omega = \frac{i}{2}\sum_p \xi_p \wedge \bar{\xi}_p,$$

$$\Omega = \xi_1 \wedge \xi_2 \wedge \xi_3.$$

It is clear that these forms are independent of the choice of such local unitary coframe, as long as it maintains the same "complex orientation." This means that the two frames can only differ by an element of SU(3) at each point on N.

We will write the Hodge star operator of N as $*_6$, the metric as g_6 and the volume form as vol_6. It is then easy to check the following identities (which will be employed often in later sections):

$$*_6^2 = (-1)^k \quad \text{on } \Omega^k(N), \qquad *_6\Omega = -i\Omega, \qquad *_6\bar{\Omega} = i\bar{\Omega},$$

$$*_6\omega = \frac{\omega^2}{2}, \qquad *_6\frac{\omega^2}{2} = \omega, \qquad (x \lrcorner \Omega) \wedge \omega = \Omega \wedge (x \lrcorner \omega). \tag{3.2}$$

The importance of SU(3)-structures for our purposes is that they naturally induce G_2-structures on $M^7 = N^6 \times L^1$, where L^1 can be $\mathbb{R}$ or S^1. Let r be a local coordinate on L^1. Then the 3-form φ defined by $\varphi = \mathrm{Re}(\Omega) - dr \wedge \omega$ is a G_2-structure on M^7, inducing the product metric $g_7 = dr^2 + g_6$ and the dual 4-form $\psi = -dr \wedge \mathrm{Im}\,\Omega - \frac{\omega^2}{2}$. See [22] for a detailed discussion of this relationship, as well as an explanation of the different sign conventions for G_2-structures. The relationships between SU(3)-structures and G_2-structures are also discussed in [9] and [8].

Definition 3.2. We can define a more general G_2-structure on M^7 which is cohomogeneity one with respect to the SU(3) action. Let $F(r)$ be a smooth, nowhere vanishing complex-valued function on L^1, and let $G(r)$ to be a smooth, everywhere positive function on L^1. Then

$$\varphi = \mathrm{Re}\big(F^3\Omega\big) - G|F|^2\, dr \wedge \omega \tag{3.3}$$

is a G_2-structure on M^7, with induced metric

$$g_7 = G^2\, dr^2 + |F|^2 g_6, \tag{3.4}$$

associated volume form

$$\mathrm{vol}_7 = G|F|^6\, dr \wedge \mathrm{vol}_6, \tag{3.5}$$

and dual 4-form

$$\psi = -G\, dr \wedge \operatorname{Im}\big(F^3 \Omega\big) - |F|^4 \frac{\omega^2}{2}. \tag{3.6}$$

With regards to the SU(3) local unitary coframe on N described in Remark 3.1, this simply corresponds to choosing $\{F \operatorname{Re}\xi_1, F\operatorname{Re}\xi_2, F\operatorname{Re}\xi_3, F\operatorname{Im}\xi_1, F\operatorname{Im}\xi_2, F\operatorname{Im}\xi_3, G\, dr\}$ to be an orthonormal G_2-adapted coframe for M^7.

Remark 3.3. We remark that the function $G(r)$ can always be set equal to 1 by defining a new local coordinate to be $\tilde{r} = \int_0^r G(s)\, ds$, so $d\tilde{r} = G(r)\, dr$. *However*, when we are considering a flow of G_2-structures $\varphi(t)$, it will be convenient to include this factor of $G(r)$, because then $G(r)$ and thus the change of variables $\tilde{r} = \tilde{r}(r)$ will in general also be t-dependent. This will become clear in Section 4.

3.1. Calabi–Yau threefolds

When both the Kähler form ω and the nonvanishing $(3,0)$ form Ω are *parallel* with respect to the Levi-Civita connection ∇ of the metric g, then (N^6, g, ω, Ω) is called a *Calabi–Yau threefold*. In particular the forms ω and Ω are both closed: $d\omega = 0$ and $d\Omega = 0$. See [20] for more about the differential geometry of Calabi–Yau manifolds. In this case, the ansatz given by Eqs. (3.3) and (3.6) for the G_2-structure on $N^6 \times L^1$ will be torsion-free (closed and coclosed) if and only if

$$d\left(\frac{1}{2}F^3\Omega + \frac{1}{2}\bar{F}^3\bar{\Omega} - GF\bar{F}\, dr \wedge \omega\right) = \frac{3}{2}\big(F^2F'\, dr \wedge \Omega + \bar{F}^2\bar{F}'\, dr \wedge \bar{\Omega}\big) = 0,$$

$$d\left(-\frac{1}{2i}GF^3\, dr \wedge \Omega + \frac{1}{2i}G\bar{F}^3\, dr \wedge \bar{\Omega} - F^2\bar{F}^2\frac{\omega^2}{2}\right) = -2\big(FF'\bar{F}^2 + F^2\bar{F}\bar{F}'\big)\, dr \wedge \frac{\omega^2}{2} = 0.$$

By comparing types, these equations are satisfied if and only if $F' = 0$. Hence F must be constant for the G_2-structure to be torsion-free. By Remark 3.3, in the time-independent case we can always assume $G = 1$, and by rescaling the SU(3)-structure on N we can assume that $F = 1$ also. Hence M^7 is then metrically a product of the Calabi–Yau 3-fold and the standard flat metric on L^1.

3.2. Nearly Kähler 6-manifolds

Another interesting SU(3)-structure that is related to G_2-geometry is that of a *nearly Kähler* 6-*manifold*. In this case, the forms ω and Ω satisfy the following system of coupled equations:

$$\begin{aligned} d\omega &= -3\operatorname{Re}(\Omega), & d\operatorname{Re}(\Omega) &= 0, \\ d\operatorname{Im}(\Omega) &= 4\frac{\omega^2}{2}, & d\frac{\omega^2}{2} &= 0. \end{aligned} \tag{3.7}$$

Of course the second column of equations in (3.7) follows immediately from the first column, but we prefer to list them all together as we will require them all for computations in Section 3.4.

An excellent survey of nearly Kähler manifolds is [26]. We remark that, other than the standard round S^6, only three other examples of compact nearly Kähler 6-manifolds are known, and these are all homogeneous spaces. The fact that these are the only compact homogeneous examples that can exist was proved by Butruille [6]. It is expected, however, that there should exist many non-homogeneous compact examples. The case of cohomogeneity-one complete nearly Kähler manifolds has been studied by Podestà–Spiro in [25] and [24].

For the purposes of the present paper, we will only need to use Eqs. (3.7) describing a nearly Kähler 6-manifold, in addition to the standard relations of an SU(3)-structure from Eqs. (3.1) and (3.2). In this case, the ansatz (3.3) and (3.6) for the G_2-structure on $N^6 \times L^1$ will be torsion-free (closed and coclosed) if and only if

$$\begin{aligned} d\varphi &= \frac{3}{2}\big(F^2F'\, dr \wedge \Omega + \bar{F}^2\bar{F}'\, dr \wedge \bar{\Omega}\big) + \frac{1}{2}F^3(4i)\frac{\omega^2}{2} + \frac{1}{2}\bar{F}^3(-4i)\frac{\omega^2}{2} + GF\bar{F}\, dr \wedge \left(-\frac{3}{2}\Omega - \frac{3}{2}\bar{\Omega}\right) \\ &= \frac{3}{2}\big(F^2F' - GF\bar{F}\big)\, dr \wedge \Omega + \frac{3}{2}\big(\bar{F}^2\bar{F}' - GF\bar{F}\big)\, dr \wedge \bar{\Omega} + 2i\big(F^3 - \bar{F}^3\big)\frac{\omega^2}{2} = 0, \end{aligned}$$

and

$$\begin{aligned} d\psi &= -2\big(FF'\bar{F}^2 + F^2\bar{F}\bar{F}'\big)\, dr \wedge \frac{\omega^2}{2} + \frac{1}{2i}GF^3\, dr \wedge (4i)\frac{\omega^2}{2} - \frac{1}{2i}G\bar{F}^3\, dr \wedge (-4i)\frac{\omega^2}{2} \\ &= \big(2G\big(F^3 + \bar{F}^3\big) - 2\big(F^2\bar{F}\bar{F}' + \bar{F}^2FF'\big)\big)\, dr \wedge \frac{\omega^2}{2} = 0. \end{aligned}$$

Again assuming that $G=1$, it is easy to check that the solution to this system of equations is $F(r)=r$, yielding the metric

$$g_7 = dr^2 + r^2 g_6$$

which is a *metric cone* over the space N^6. Here we need to take $L^1=(0,\infty)$. In fact, one can also *define* nearly Kähler 6-manifolds to be exactly those spaces for which the Riemannian cone over them has holonomy contained in G_2. See Bär [1] for details.

Remark 3.4. See also Cleyton–Swann [11] for another application of SU(3)-structures to cohomogeneity-one G_2-structures.

3.3. *Some invariant formulas on $M^7 = N^6 \times L^1$*

In this section we collect together some formulas involving the Hodge star operators $*_6$ and $*_7$ on N^6 and M^7, respectively, which we will use in both the Calabi–Yau and the nearly Kähler cases to study the Laplacian co-flow. We also discuss the Laplacian and gradient for functions on M^7 which depend only on the coordinate r on L^1, which we will need later to express the evolution and soliton equations in an invariant form.

We consider the ansatz (3.3) for a cohomogeneity-one G_2-structure on M^7. To simplify the calculations somewhat, we will sometimes write

$$F = he^{i\theta}$$

for some smooth *real-valued* functions h and θ on L^1. Hence we can write Eqs. (3.3) and (3.6) as

$$\begin{aligned}\varphi &= \frac{F^3}{2}\Omega + \frac{\bar{F}^3}{2}\bar{\Omega} - Gh^2\, dr\wedge\omega,\\ \psi &= \frac{iGF^3}{2}\, dr\wedge\Omega - \frac{iG\bar{F}^3}{2}\, dr\wedge\bar{\Omega} - h^4\frac{\omega^2}{2},\end{aligned} \tag{3.8}$$

and the metric and volume form as

$$g_7 = G^2\, dr^2 + h^2 g_6, \qquad \mathrm{vol}_7 = Gh^6\, dr\wedge \mathrm{vol}_6. \tag{3.9}$$

Using (3.9) for the metric and the volume form on M^7, it is easy to see that if α is any k-form on N^6, then we have

$$\begin{aligned} *_7\alpha &= (-1)^k h^{6-2k} G\, dr\wedge *_6\alpha,\\ *_7(dr\wedge\alpha) &= h^{6-2k}G^{-1} *_6 \alpha. \end{aligned} \tag{3.10}$$

Using these equations and (3.2), we find that

$$\left.\begin{aligned} *_7\omega &= h^2 G\, dr\wedge\frac{\omega^2}{2}, & *_7(dr\wedge\omega) &= h^2G^{-1}\frac{\omega^2}{2},\\ *_7\Omega &= iG\, dr\wedge\Omega, & *_7(dr\wedge\Omega) &= -iG^{-1}\Omega,\\ *_7\left(\frac{\omega^2}{2}\right) &= h^{-2}G\, dr\wedge\omega, & *_7\left(dr\wedge\frac{\omega^2}{2}\right) &= h^{-2}G^{-1}\omega. \end{aligned}\right\} \tag{3.11}$$

Remark 3.5. Throughout this paper, we will always use a prime $'$ to denote differentiation with respect to the coordinate r on L^1.

Suppose that $f=f(r)$ is a function depending only on the coordinate r on L^1. Then using (3.10) we can compute that its Hodge Laplacian $\Delta_d f$ is given by

$$\begin{aligned}\Delta_d f = d^*df &= -*_7\, d *_7 f'\, dr = -*_7\, d\big(f' *_7 dr\big) = -*_7\, d\left(\frac{f'h^6}{G}\mathrm{vol}_6\right)\\ &= -*_7\left(\left(\frac{f'h^6}{G}\right)' dr\wedge \mathrm{vol}_6\right) = -\frac{1}{Gh^6}\left(\frac{f'h^6}{G}\right)'\end{aligned}$$

Remark 3.6. We will use the symbol Δ (without the d subscript) to denote the *rough Laplacian* $\nabla^*\nabla$, which, on functions, differs from Δ_d by a sign.

Hence the above equation gives

$$\Delta f = \frac{1}{Gh^6}\left(\frac{f'h^6}{G}\right)' = \frac{f''}{G^2} + \frac{6h'f'}{hG^2} - \frac{f'G'}{G^3}. \tag{3.12}$$

We also note by (3.9), if $f = f(r)$ and $\rho = \rho(r)$, and ∇ denotes the *gradient* with respect to g_7, then we have that

$$\langle \nabla f, \nabla \rho \rangle = \langle df, d\rho \rangle = f'\rho' \langle dr, dr \rangle = \frac{f'\rho'}{G^2}, \qquad |\nabla f|^2 = \frac{(f')^2}{G^2}. \tag{3.13}$$

3.4. The torsion forms

In this section we compute the four torsion forms τ_0, τ_1, τ_2, and τ_3 that we defined in Section 2 for our cohomogeneity-one G_2-structure on $N^6 \times L^1$, in the two cases where N^6 is either Calabi–Yau or nearly Kähler. Differentiating the forms in (3.8) gives

$$d\varphi = \frac{(F^3)'}{2} dr \wedge \Omega + \frac{(\bar{F}^3)'}{2} dr \wedge \bar{\Omega} + Gh^2\, dr \wedge d\omega + \frac{F^3}{2} d\Omega + \frac{\bar{F}^3}{2} d\bar{\Omega},$$
$$d\psi = -\frac{iGF^3}{2} dr \wedge d\Omega + \frac{iG\bar{F}^3}{2} dr \wedge d\bar{\Omega} - \left(h^4\right)' dr \wedge \frac{\omega^2}{2} - h^4 d\left(\frac{\omega^2}{2}\right).$$

In the Calabi–Yau case, we have $d\omega = 0$ and $d\Omega = 0$, while in the nearly Kähler case, Eqs. (3.7) say

$$d\omega = -\frac{3}{2}\Omega - \frac{3}{2}\bar{\Omega}, \qquad d\left(\frac{\omega^2}{2}\right) = 0, \qquad d\Omega = 4i\frac{\omega^2}{2}. \tag{3.14}$$

Therefore,

$$\left.\begin{aligned}
&\text{when } N^6 \text{ is Calabi–Yau:}\\
&\quad d\varphi = \frac{(F^3)'}{2} dr \wedge \Omega + \frac{(\bar{F}^3)'}{2} dr \wedge \bar{\Omega},\\
&\quad d\psi = -\left(h^4\right)' dr \wedge \frac{\omega^2}{2};\\
&\text{when } N^6 \text{ is nearly Kähler:}\\
&\quad d\varphi = \left(\frac{(F^3)'}{2} - \frac{3}{2}Gh^2\right) dr \wedge \Omega + \left(\frac{(\bar{F}^3)'}{2} - \frac{3}{2}Gh^2\right) dr \wedge \bar{\Omega} + 2i\left(F^3 - \bar{F}^3\right)\frac{\omega^2}{2},\\
&\quad d\psi = \left(2G\left(F^3 + \bar{F}^3\right) - \left(h^4\right)'\right) dr \wedge \frac{\omega^2}{2}.
\end{aligned}\right\} \tag{3.15}$$

Using the identities of (3.11) we immediately get

$$\left.\begin{aligned}
&\text{when } N^6 \text{ is Calabi–Yau:}\\
&\quad *_7(d\varphi) = -\frac{i(F^3)'}{2G}\Omega + \frac{i(\bar{F}^3)'}{2G}\bar{\Omega},\\
&\quad *_7(d\psi) = -\frac{(h^4)'}{Gh^2}\omega;\\
&\text{when } N^6 \text{ is nearly Kähler:}\\
&\quad *_7(d\varphi) = -\frac{i}{G}\left(\frac{(F^3)'}{2} - \frac{3}{2}Gh^2\right)\Omega + \frac{i}{G}\left(\frac{(\bar{F}^3)'}{2} - \frac{3}{2}Gh^2\right)\bar{\Omega} + \frac{2iG}{h^2}\left(F^3 - \bar{F}^3\right) dr \wedge \omega,\\
&\quad *_7(d\psi) = \frac{1}{Gh^2}\left(2G\left(F^3 + \bar{F}^3\right) - \left(h^4\right)'\right)\omega.
\end{aligned}\right\} \tag{3.16}$$

We are now in a position to compute the torsion forms of these G_2-structures.

Lemma 3.7. *For such a G_2-structure, the zero-torsion τ_0 and the one-torsion τ_1 are as follows*:

$$\left.\begin{aligned}
&\textit{when } N^6 \textit{ is Calabi–Yau}: && \tau_0 = \frac{12}{7G}\theta', && \tau_1 = d(\log h);\\
&\textit{when } N^6 \textit{ is nearly Kähler}: && \tau_0 = \frac{12}{7}\left(\frac{\theta'}{G} + \frac{2\sin 3\theta}{h}\right), && \tau_1 = \left(\frac{h' - G\cos 3\theta}{h}\right) dr
\end{aligned}\right\} \tag{3.17}$$

and the two-torsion τ_2 *always vanishes*:

$$\tau_2 = 0 \tag{3.18}$$

in both the Calabi–Yau and the nearly Kähler cases.

Proof. Substitute Eqs. (3.15) and (3.16) into the formulas (2.2) and (2.3) for the zero-torsion τ_0 and the one-torsion τ_1, and use (3.11). It is a tedious but straightforward computation to obtain (3.17). Now Eqs. (2.1) can be solved for the two-torsion τ_2 and the three-torsion τ_3:

$$\tau_2 = *_7(d\psi) - 4 *_7 (\tau_1 \wedge \psi),$$

$$\tau_3 = *_7(d\varphi) - \tau_0\varphi - 3 *_7 (\tau_1 \wedge \varphi).$$

From these we can obtain an explicit (albeit complicated) formula for τ_3, which we omit here because we will not require it in the present paper. The result of the computation for τ_2 is that, in *both* the Calabi–Yau and the nearly Kähler cases, $\tau_2 = 0$. □

Remark 3.8. The torsion forms for a G_2-structure that is a warped product over a nearly Kähler 6-manifold have previously appeared in Cleyton–Ivanov [10]. The authors thank Sergey Grigorian for alerting them to this fact.

The fact that these G_2-structures always have vanishing two-torsion τ_2 for any h and θ will be useful later. Note that this is in stark contract to the closed G_2-structures as studied in [5,4,28,29] where τ_2 is the *only* nonvanishing torsion form. It is for this reason that the sensible flow of G_2-structures with such an SU(3) symmetry to consider this the Laplacian *co-flow* which we discuss in Section 4.

4. The Laplacian co-flow of G_2-structures

In this section we introduce the Laplacian *co-flow* of a coclosed G_2-structure and discuss some of its general properties, including its soliton solutions. Then we concentrate specifically on the G_2-structures (3.8) arising from a warped product of 1-manifold L^1 with a Calabi–Yau or a nearly Kähler 6-manifold N^6.

Definition 4.1. We say that a time-dependent G_2-structure $\varphi = \varphi(t)$ on a 7-manifold M^7, defined for t in some interval $[0, T)$, satisfies the *Laplacian co-flow equation* if for all times t for which $\varphi(t)$ is defined, we have

$$\frac{\partial \psi}{\partial t} = -\Delta_d \psi, \tag{4.1}$$

where $\psi(t) = *_{\varphi(t)}\varphi(t)$ is the Hodge dual 4-form of $\varphi(t)$ and $\Delta_d = dd^* + d^*d$ is the Hodge Laplacian with respect to the metric $g(t) = g_{\varphi(t)}$.

In this paper, we will assume that this flow has short-time existence and uniqueness if we start with an initially coclosed G_2-structure. This is very likely, since the flow is qualitatively very similar to the Laplacian flow $\frac{\partial\varphi}{\partial t} = -\Delta_d\varphi$ which does have short-time existence and uniqueness for an initially closed G_2-structure. Also, entirely analogous to the fact that the Laplacian flow $\frac{\partial\varphi}{\partial t} = -\Delta_d\varphi$ preserves the closed condition, the Laplacian co-flow $\frac{\partial\psi}{\partial t} = -\Delta_d\psi$ will preserve the coclosed condition. See [5,4,29] for these results in the case of the Laplacian flow. The main goal of the present paper, in any case, is to study the soliton solutions to this flow in certain particular situations with symmetry.

Remark 4.2. By Eqs. (2.1), a G_2-structure is coclosed exactly when $\tau_1 = 0$ and $\tau_2 = 0$.

4.1. Soliton solutions

As with the Ricci flow (and other geometric flows), it is of interest to consider "self-similar solutions" which are evolving by diffeomorphisms and scalings. If f_t is a 1-parameter family of diffeomorphisms generated by a vector field X on M, and if $c(t) = 1 + \lambda t$, then differentiation shows that a coclosed G_2-structure $\varphi(t) = c(t) f_t^*\varphi(t)$ is a solution to the co-flow (4.1) if and only if

$$-\Delta_d\psi = \mathcal{L}_X\psi + \lambda\psi = d(X \lrcorner \psi) + \lambda\psi \tag{4.2}$$

using the fact that $d\psi = 0$. In particular, a *gradient co-flow soliton* is a solution (4.2) where $X = \nabla k$ for some C^2 function k on M. As in the case of Ricci flow, we say that the soliton is *expanding*, *steady*, or *shrinking* if λ is positive, zero, or negative, respectively.

Proposition 4.3. *If M^7 is compact, then there are no expanding or steady soliton solutions of (4.2), other than the trivial case of a torsion-free G_2-structure in the steady case.*

Proof. We take the wedge product of both sides of (4.2) with $\varphi = *_\psi$ and integrate over M to obtain

$$\int_M \langle \Delta_d \psi, \psi \rangle \mathrm{vol} + \lambda \int_M |\psi|^2 \mathrm{vol} + \int_M \langle d(X \lrcorner \psi), \psi \rangle \mathrm{vol} = 0. \tag{4.3}$$

Since M is compact, we have

$$\int_M \langle d(X \lrcorner \psi), \psi \rangle \mathrm{vol} = \int_M \langle X \lrcorner \psi, d^* \psi \rangle \mathrm{vol}.$$

But the G_2-structure is coclosed, so $\tau_1 = 0$ and hence $d^*\psi = *d * \psi = *d\varphi = *(\tau_0 \psi + *\tau_3) = \tau_0 \varphi + \tau_3$. Therefore $d^*\psi$ lies in the space $\Lambda^4_1 \oplus \Lambda^4_{27}$, while $X \lrcorner \psi$ lies in Λ^4_7. Since this decomposition of Λ^4 is pointwise orthogonal with respect to the metric g_φ, we see that the last term in (4.3) vanishes. Since $|\psi|^2 = 7$, we get

$$\langle\langle \Delta_d \psi, \psi \rangle\rangle + 7\lambda \int_M \mathrm{vol} = \|d^* \psi\|^2 + 7\lambda \, \mathrm{Vol}(M) = 0,$$

again using the fact that $d\psi = 0$. Thus we cannot have $\lambda > 0$, and if $\lambda = 0$ then the G_2-structure must be torsion-free. In the latter case X must be a vector field generating a G_2-symmetry: $\mathcal{L}_X \psi = 0$. Since M is compact, there will be no such nonzero X unless M has reducible holonomy (see [20], for example). □

Remark 4.4. It is easy to find nontrivial examples of compact shrinking solitons: a *nearly* G_2-structure is one for which $d\psi = 0$ and $d\varphi = \mu \psi$ for some nonzero constant μ. In this case $\Delta_d \psi = \mu^2 \psi$, and these give examples of compact shrinking solitons with $X = 0$ and $\lambda = -\mu^2$. Nearly G_2 manifolds are those for which the metric cone over them has Spin(7) holonomy. There are many known compact examples. See [1] or [17] for more about nearly G_2 manifolds.

Remark 4.5. A very similar argument as in Proposition 4.3 can be used to show that for the Laplacian flow $\frac{\partial \varphi}{\partial t} = -\Delta_d \varphi$ of *closed* G_2-structures, in the compact case there are no expanding or steady solitons, other than the trivial case of a torsion-free G_2-structure when $\lambda = 0$. The nearly G_2 manifolds are examples of compact shrinking solitons for this flow as well.

For the cohomogeneity-one G_2-structures that we consider in this paper, the only natural (with respect to the symmetry of the structure) vector fields must be of gradient type, so we will need only consider such gradient solitons, which are C^2 solutions $\psi(t)$ to

$$-\Delta_d \psi = \mathcal{L}_{\nabla k} \psi + \lambda \psi \tag{4.4}$$

for some C^2 function k on M and some constant λ. Also, for the examples we consider, $M^7 = N^6 \times L^1$, and while N^6 will always be taken to be compact, we can have either $L^1 \cong S^1$ or $L^1 \cong \mathbb{R}$, so we will not always be able to use Proposition 4.3.

4.2. The Hodge Laplacian on $M^7 = N^6 \times L^1$

In this section we derive explicitly the Hodge Laplacian $-\Delta_d \psi$ for the G_2-structures (3.8) with SU(3) symmetry when N^6 is Calabi–Yau or nearly Kähler. Recall that we consider only coclosed G_2-structures of these types. By Lemma 3.7, the two-torsion τ_2 is always zero, but we need to *impose* the condition that $\tau_1 = 0$, which, as we noted above, will be preserved under the Laplacian co-flow $\frac{\partial \psi}{\partial t} = -\Delta_d \psi$.

Assumption 4.6. The G_2-structure (3.8) is assumed to be coclosed. Thus $\tau_1 = 0$. By Lemma 3.7, this means that we assume:

$$\left.\begin{array}{ll} \text{when } N^6 \text{ is Calabi–Yau:} & h' = 0; \\ \text{when } N^6 \text{ is nearly Kähler:} & h' = G\cos 3\theta. \end{array}\right\} \tag{4.5}$$

With this assumption, it is easy to compute $-\Delta_d \psi = -dd^*\psi = -d *_7 d\varphi$.

Lemma 4.7. *For these G_2-structures, we have that*

$$\left.\begin{aligned} &\textit{when } N^6 \textit{ is Calabi–Yau}: \quad -\Delta_d\psi = \left(\frac{i(F^3)'}{2G}\right)' dr\wedge\Omega + \left(-\frac{i(\bar F^3)'}{2G}\right)' dr\wedge\bar\Omega; \\ &\textit{when } N^6 \textit{ is nearly Kähler}: \quad -\Delta_d\psi = A\,dr\wedge\Omega + \bar A\,dr\wedge\bar\Omega + B\frac{\omega^2}{2}, \\ &\quad \textit{where } A = \left[\left(\frac{i(F^3)'}{2G} - \frac{3i}{2}h^2\right)' + 6Gh\sin 3\theta\right] \textit{ and } B = \left[-\frac{4}{G}\left(h^3\cos 3\theta\right)' + 12h^2\right]. \end{aligned}\right\} \tag{4.6}$$

Proof. We use the expression for $*_7(d\varphi)$ that we derived in (3.16). In the Calabi–Yau case, we have

$$-\Delta_d\psi = -d*_7(d\varphi) = \left(\frac{i(F^3)'}{2G}\right)' dr\wedge\Omega + \left(-\frac{i(\bar F^3)'}{2G}\right)' dr\wedge\bar\Omega$$

since $d\Omega = 0$ and $d\bar\Omega = 0$. This establishes the first half of (4.6). In the nearly Kähler case, we use also (3.14) to obtain

$$\begin{aligned} -\Delta_d\psi = -d*_7(d\varphi) &= \left(\frac{i(F^3)'}{2G} - \frac{3i}{2}h^2\right)' dr\wedge\Omega + \left(\frac{-i(F^3)'}{2G} + \frac{3i}{2}h^2\right)' dr\wedge\bar\Omega \\ &\quad + \left(\frac{i(F^3)'}{2G} - \frac{3i}{2}h^2\right) d\Omega + \left(\frac{-i(\bar F^3)'}{2G} + \frac{3i}{2}h^2\right) d\bar\Omega + \frac{2iG}{h^2}\left(F^3 - \bar F^3\right) dr\wedge d\omega \\ &= \left[\left(\frac{i(F^3)'}{2G} - \frac{3i}{2}h^2\right)' - \frac{3iG}{h^2}\left(F^3 - \bar F^3\right)\right] dr\wedge\Omega \\ &\quad + \left[\left(\frac{-i(F^3)'}{2G} + \frac{3i}{2}h^2\right)' - \frac{3iG}{h^2}\left(F^3 - \bar F^3\right)\right] dr\wedge\bar\Omega \\ &\quad + \left[4i\left(\frac{i(F^3)'}{2G} - \frac{3i}{2}h^2\right) - 4i\left(\frac{-i(F^3)'}{2G} + \frac{3i}{2}h^2\right)\right]\frac{\omega^2}{2}. \end{aligned}$$

Using $F = he^{i\theta}$, this expression simplifies to

$$\begin{aligned} -\Delta_d\psi &= \left[\left(\frac{i(F^3)'}{2G} - \frac{3i}{2}h^2\right)' + 6Gh\sin 3\theta\right] dr\wedge\Omega \\ &\quad + \left[\left(\frac{-i(F^3)'}{2G} + \frac{3i}{2}h^2\right)' + 6Gh\sin 3\theta\right] dr\wedge\bar\Omega \\ &\quad + \left[-\frac{4}{G}\left(h^3\cos 3\theta\right)' + 12h^2\right]\frac{\omega^2}{2}. \end{aligned}$$

which establishes the second half of (4.6). □

Recall that we have

$$\psi = \frac{iGF^3}{2}dr\wedge\Omega - \frac{iG\bar F^3}{2}dr\wedge\bar\Omega - h^4\frac{\omega^2}{2}. \tag{4.7}$$

In the Laplacian co-flow $\frac{\partial\psi}{\partial t} = -\Delta_d\psi$, only the functions $F = he^{i\theta}$ and G depend on t and the coordinate r on L^1. We are now ready to study the co-flow and corresponding soliton equations in detail for the Calabi–Yau and the nearly Kähler cases in the next two sections.

5. The case when N^6 is Calabi–Yau

We begin with the evolution equations.

5.1. The CY evolution equations

Theorem 5.1. *Let N^6 be Calabi–Yau, and let $M^7 = N^6 \times L^1$ be a manifold with* coclosed G_2*-structure given by (3.8), with $d\psi = 0$. Then under the Laplacian co-flow $\frac{\partial\psi}{\partial t} = -\Delta_d\psi$, the functions $F = he^{i\theta}$ and G on L^1 (depending also on the time parameter t) satisfy the following evolution equations.*

$$h = 1, \qquad \frac{\partial\theta}{\partial t} = \Delta\theta, \qquad \frac{\partial G}{\partial t} = -9G|\nabla\theta|^2, \tag{5.1}$$

where the rough Laplacian Δ, the gradient ∇, and the pointwise norm $|\cdot|$ are all taken with respect to the metric $g_7 = G^2\,dr^2 + h^6 g_6$ on M^7.

Proof. Differentiating (4.7) with respect to t and using Lemma 4.7, we can compute $\frac{\partial \psi}{\partial t} = -\Delta_d \psi$ and equate the coefficients of $dr \wedge \Omega$, $dr \wedge \bar{\Omega}$, and $\frac{\omega^2}{2}$. We find that

$$\frac{\partial}{\partial t}\left(\frac{iGF^3}{2}\right) = \left(\frac{i(F^3)'}{2G}\right)' \quad \text{and} \quad \frac{\partial}{\partial t}\left(-h^4\right) = 0.$$

The second equation says that $\frac{\partial h}{\partial t} = 0$, so that h is constant in time as well. (Recall from (4.5) that the condition $\tau_1 = 0$ in this case was that h is also independent of r.) Without loss of generality, by rescaling the metric on the Calabi–Yau manifold N^6, we can assume that $h = 1$ from now on. Substituting $h = 1$ into the first equation above and simplifying, we obtain

$$\frac{\partial}{\partial t}\left(Ge^{i3\theta}\right) = \left(\frac{(e^{i3\theta})'}{G}\right)'.$$

Expanding and simplifying, we have

$$\left(\frac{\partial G}{\partial t} + 3iG\frac{\partial \theta}{\partial t}\right)e^{i3\theta} = \left(\frac{3i\theta'}{G}e^{i3\theta}\right)' = \left(-\frac{3iG'\theta'}{G^2} + \frac{3i\theta''}{G} - \frac{9(\theta')^2}{G}\right)e^{3i\theta}.$$

Equating real and imaginary parts gives

$$\frac{\partial G}{\partial t} = -\frac{9(\theta')^2}{G}, \qquad \frac{\partial \theta}{\partial t} = \frac{\theta''}{G^2} - \frac{G'\theta'}{G^3}.$$

Since in this case we have $h = 1$, Eqs. (3.12) and (3.13) give that the above equations can be invariantly expressed as

$$\frac{\partial G}{\partial t} = -9G|\nabla\theta|^2, \qquad \frac{\partial \theta}{\partial t} = \Delta\theta,$$

which is what we wanted to prove. $\square$

Note that even though the phase function $\theta(r,t)$ satisfies what appears to be a simple heat equation, the Laplacian Δ is taken with respect to the metric (3.4) that is changing with time. This makes it very difficult to establish long-time existence without a much more delicate analysis. In general, we expect that there should be singularity formation in finite time, as is the case with most geometric evolution equations.

5.2. The CY soliton equations

Next, we turn to the *soliton* equations in this case. Since this is a time-static situation, we can without loss of generality reparametrize the local coordinate r so that $G = 1$, as discussed in Remark 3.3. We are looking for soliton solutions which have the same SU(3) symmetry as the evolution equations, so the only possible vector fields are of the form $X = s(r)\frac{\partial}{\partial r}$ for some function $s = s(r)$ on L^1. By letting $k(r) = \int_{r_0}^{r} s(u)\,du$ be an antiderivative, we can assume that $X = \nabla k = k'\frac{\partial}{\partial r}$ is a gradient vector field for some function $k = k(r)$ on L^1. Note that since $G = 1$, we do indeed have $(dr)^\sharp = \frac{\partial}{\partial r}$ so this is the correct expression for ∇k. The soliton equation, as derived in (4.4), is

$$-\Delta_d \psi = \mathcal{L}_{\nabla k}\psi + \lambda\psi = d(\nabla k \lrcorner \psi) + \lambda\psi \tag{5.2}$$

since $d\psi = 0$.

Theorem 5.2. *The coclosed* G_2*-structures which satisfy the soliton equation* (5.2) *when* N^6 *is Calabi–Yau are given by* (3.8) *where* $G = 1$ *and*

$$\lambda = 0, \qquad h = 1, \qquad \theta = \frac{2}{3}\arctan\left(ce^{br}\right), \qquad X = b\left(\frac{1 - c^2r^{2br}}{1 + c^2e^{2br}}\right)\frac{\partial}{\partial r},$$

for some real constants b and c. In particular, all the soliton solutions are steady *and the only solutions which exist in the case that* $L^1 \cong S^1$ *is* compact *are constant θ and k' (corresponding to $b = 0$ or $c = 0$) which are trivial translations and phase rotations of the standard torsion-free* G_2*-structure on* $N^6 \times S^1$*. However, in the case where* $L^1 \cong \mathbb{R}$ *is* noncompact*, we do obtain nontrivial soliton solutions on* $N^6 \times \mathbb{R}$*.*

Proof. We compute using (4.7) and $G = 1$ that

$$d(\nabla k \lrcorner \psi) = d\left(k'\frac{\partial}{\partial r}\lrcorner\psi\right) = d\left(\frac{iF^3k'}{2}\Omega - \frac{i\bar{F}^3k'}{2}\bar{\Omega}\right) = \frac{i}{2}\left(F^3k'\right)'dr \wedge \Omega - \frac{i}{2}\left(\bar{F}^3k'\right)'dr \wedge \bar{\Omega}.$$

Substituting the above expression into (5.2) and using (4.7) and (4.6), and comparing coefficients, we have

$$\frac{i}{2}\left(F^3\right)'' = \lambda\frac{iF^3}{2} + \frac{i}{2}\left(F^3 k'\right)', \qquad 0 = -\lambda h^4.$$

Since $h > 0$, the second equation says $\lambda = 0$. That is, there are *only* steady solitons in this case. Comparing with Remark 4.4, this implies (at least if L^1 is compact) that this G_2-structure cannot be nearly G_2. Indeed, it is easy to check directly that for this ansatz, the three-torsion τ_3 will vanish only when τ_0 also vanishes and φ is completely torsion-free. Now with $\lambda = 0$, and recalling that $h = 1$, the first equation above simplifies to

$$\left(e^{i3\theta}\right)'' - \left(e^{i3\theta}k'\right)' = 0,$$

which can be immediately integrated once to yield

$$\left(e^{i3\theta}\right)' - e^{i3\theta}k' = -b = -(b_1 + ib_2)$$

for some constant $b \in \mathbb{C}$. Taking real and imaginary parts, we get

$$(\cos 3\theta)' - (\cos 3\theta)k' = -b_1, \qquad (\sin 3\theta)' - (\sin 3\theta)k' = -b_2. \tag{5.3}$$

In (5.3), if we multiply the first equation by $\sin 3\theta$ and the second equation by $\cos 3\theta$ and subtract, we eliminate k' and obtain

$$\begin{aligned} -b_1 \sin 3\theta + b_2 \cos 3\theta &= (\sin 3\theta)(\cos 3\theta)' - (\cos 3\theta)(\sin 3\theta)' \\ &= -3\theta' \sin^2 3\theta - 3\theta' \cos^2 3\theta = -3\theta' \end{aligned}$$

and thus

$$3\theta' = b_1 \sin 3\theta - b_2 \cos 3\theta. \tag{5.4}$$

But in (5.3), we can also multiply the first equation by $\cos 3\theta$ and the second equation by $\sin 3\theta$ and add, and we find that

$$-b_1 \cos 3\theta - b_2 \sin 3\theta = (\cos 3\theta)(\cos 3\theta)' + (\sin 3\theta)(\sin 3\theta)' - \left(\cos^2 3\theta\right)k' - \left(\sin^2 3\theta\right)k' = -k'$$

and therefore

$$k' = b_1 \cos 3\theta + b_2 \sin 3\theta. \tag{5.5}$$

Eq. (5.4) can actually be integrated exactly, although the solution is quite complicated for general $b \in \mathbb{C}$. However, given any values $\theta(r_o)$ and $k'(r_o)$ of the functions θ and k' at some fixed point $r_o \in L^1$, we see that by performing a "rotation" of the Calabi–Yau holomorphic volume form $\Omega \mapsto e^{i\gamma}\Omega$ for an appropriate constant γ, we can arrange that $b_2 = 0$ so $b = b_1$ is purely real. We are always free to do such a rotation because the holomorphic volume form Ω of a Calabi–Yau manifold is only defined up to a constant phase factor. Then Eq. (5.4) becomes

$$\frac{3\,d\theta}{\sin 3\theta} = b\,dr$$

which has solution

$$\theta(r) = \frac{2}{3}\arctan\left(ce^{br}\right)$$

for some real constants b and c depending on the "initial" conditions. This can then be substituted into (5.5) to directly solve for k'. We have

$$k' = b\cos\left(2\arctan\left(ce^{br}\right)\right) = b\left(\frac{1 - c^2 r^{2br}}{1 + c^2 e^{2br}}\right),$$

and the proof is complete. □

We remark that since $h = 1$ and $G = 1$ for these soliton solutions, the metric (3.4) on M^7 is just the product of the flat metric on L^1 and the Calabi–Yau metric on N^6. While the metric in this case is not new, the corresponding G_2-structure φ is in general *not* torsion-free. Indeed, from Lemma 3.7 we see that τ_0 will not vanish unless θ is constant. This is similar, but slightly different, to the fact that the standard Euclidean metric on $\mathbb{R}^n$ can be written in a nontrivial way as a gradient Ricci soliton.

6. The case when N^6 is nearly Kähler

Now suppose that N^6 is nearly Kähler. Again we begin with the evolution equations.

6.1. The NK evolution equations

Theorem 6.1. *Let N^6 be nearly Kähler, and let $M^7 = N^6 \times L^1$ be a manifold with* coclosed G_2*-structure given by (3.8), with $d\psi = 0$. Then under the Laplacian co-flow $\frac{\partial \psi}{\partial t} = -\Delta_d \psi$, the functions $F = he^{i\theta}$ and G on L^1 (depending also on the time parameter t) satisfy the following evolution equations.*

$$\frac{\partial h}{\partial t} = \Delta h - \frac{3}{h}\left(1 + |\nabla h|^2\right), \qquad \frac{\partial \theta}{\partial t} = \Delta \theta - \frac{\sin 6\theta}{h^2}, \qquad \frac{\partial G}{\partial t} = -\left(9|\nabla \theta|^2 + 3\left|\frac{\sin 3\theta}{h}\right|^2\right)G, \tag{6.1}$$

where the rough Laplacian Δ, the gradient ∇, and the pointwise norm $|\cdot|$ are all taken with respect to the metric $g_7 = G^2\,dr^2 + h^6 g_6$ on M^7.

Proof. Differentiating (4.7) with respect to t and using Lemma 4.7, we can compute $\frac{\partial \psi}{\partial t} = -\Delta_d \psi$ and equate the coefficients of $dr \wedge \Omega$, $dr \wedge \bar{\Omega}$, and $\frac{\omega^2}{2}$. We find that

$$\frac{\partial}{\partial t}\left(\frac{iGF^3}{2}\right) = \left(\frac{i(F^3)'}{2G} - \frac{3i}{2}h^2\right)' + 6Gh\sin 3\theta \quad \text{and} \quad \frac{\partial}{\partial t}\left(-h^4\right) = -\frac{4}{G}\left(h^3\cos 3\theta\right)' + 12h^2. \tag{6.2}$$

The first equation is a complex equation, and can be simplified to

$$\frac{\partial}{\partial t}\left(GF^3\right) = \left(\frac{(F^3)'}{G} - 3h^2\right)' - 12iGh\sin 3\theta. \tag{6.3}$$

The second equation is a real equation and can be simplified to

$$\frac{\partial h}{\partial t} = \frac{(h^3\cos 3\theta)'}{Gh^3} - \frac{3}{h}. \tag{6.4}$$

Recall, however, that we *also* have the $\tau_1 = 0$ condition from (4.5) that says

$$h' = G\cos 3\theta. \tag{6.5}$$

Now at first glance it would appear that this system is overdetermined, because we have four equations for three functions G, h, and θ. However, we will now see that there is indeed some redundancy. The real part of (6.3) is

$$\frac{\partial}{\partial t}\left(Gh^3\cos 3\theta\right) = \left[\frac{(h^3\cos 3\theta)'}{G} - 3h^2\right]'.$$

If we substitute (6.5) into the left-hand side of the above expression, we obtain

$$\frac{\partial}{\partial t}\left(h^3 h'\right) = \frac{\partial}{\partial t}\left(\frac{h^4}{4}\right)' = \left[\frac{(h^3\cos 3\theta)'}{G} - 3h^2\right]'$$

which, up to a factor of (-4), is exactly the derivative with respect to r of the second equation in (6.2) which led to (6.4). Thus, the independent equations are (6.4) and (6.5) and the imaginary part of (6.3):

$$\frac{\partial}{\partial t}\left(Gh^3\sin 3\theta\right) = \left(\frac{(h^3\sin 3\theta)'}{G}\right)' - 12Gh\sin 3\theta. \tag{6.6}$$

We need to extract invariant expressions for the time derivatives of G, h, and θ from the above equations. We begin by substituting (6.5) into (6.4) to eliminate $\cos 3\theta$:

$$\frac{\partial h}{\partial t} = \frac{1}{Gh^3}\left(\frac{h^3 h'}{G}\right)' - \frac{3}{h} = \frac{1}{Gh^3}\left[\frac{3h^2(h')^2}{G} + \frac{h^3 h''}{G} - \frac{h^3 h' G'}{G^2}\right] - \frac{3}{h},$$

$$\frac{\partial h}{\partial t} = \frac{h''}{G^2} + \frac{3(h')^2}{hG^2} - \frac{h'G'}{G^3} - \frac{3}{h}. \tag{6.7}$$

The above form of $\frac{\partial h}{\partial t}$ will be useful later. We can further simplify it as

$$\frac{\partial h}{\partial t} = \left(\frac{h''}{G^2} + \frac{6(h')^2}{hG^2} - \frac{h'G'}{G^3}\right) - \frac{3}{h} - \frac{3(h')^2}{hG^2} = \Delta h - \frac{3}{h}\left(1 + |\nabla h|^2\right),$$

where we have used (3.12) and (3.13). This proves the first part of (6.1). We will need to work a bit harder to get the evolution equations for θ and G. Let S denote the right-hand side of (6.6):

$$S = \left(\frac{(h^3 \sin 3\theta)'}{G}\right)' - 12Gh \sin 3\theta. \tag{6.8}$$

Expanding the left-hand side of (6.6) and rearranging, we find

$$\left(h^3 \sin 3\theta\right)\frac{\partial G}{\partial t} + \left(3Gh^3 \cos 3\theta\right)\frac{\partial \theta}{\partial t} = S - \left(3Gh^2 \sin 3\theta\right)\frac{\partial h}{\partial t}. \tag{6.9}$$

This equation is linear in $\frac{\partial G}{\partial t}$ and $\frac{\partial \theta}{\partial t}$. We can get another one by differentiating (6.5) with respect to t:

$$(\cos 3\theta)\frac{\partial G}{\partial t} - (3G \sin 3\theta)\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial t}h' = \left(\frac{\partial h}{\partial t}\right)'.$$

Dividing (6.9) by h^3, we now have the following system of linear equations:

$$\begin{pmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{pmatrix}\begin{pmatrix} 3G\frac{\partial \theta}{\partial t} \\ \frac{\partial G}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{S}{h^3} - \frac{3G \sin 3\theta}{h}\frac{\partial h}{\partial t} \\ (\frac{\partial h}{\partial t})' \end{pmatrix}.$$

This system is easily solved to yield

$$3G\frac{\partial \theta}{\partial t} = (\cos 3\theta)\left(\frac{S}{h^3} - \frac{3G \sin 3\theta}{h}\frac{\partial h}{\partial t}\right) - \sin 3\theta\left(\frac{\partial h}{\partial t}\right)',$$

$$\frac{\partial G}{\partial t} = (\sin 3\theta)\left(\frac{S}{h^3} - \frac{3G \sin 3\theta}{h}\frac{\partial h}{\partial t}\right) + \cos 3\theta\left(\frac{\partial h}{\partial t}\right)'.$$

We can now substitute the expression (6.8) for S, the expression (6.7) for $\frac{\partial h}{\partial t}$, and the derivative with respect to r of (6.7) for $(\frac{\partial h}{\partial t})'$ into the above equations. We also repeatedly use (6.5) to eliminate all terms involving h' at every stage. After much computation, the result is:

$$\frac{\partial G}{\partial t} = -\frac{3G \sin^2 3\theta}{h^2} - \frac{9(\theta')^2}{G}, \tag{6.10}$$

$$\frac{\partial \theta}{\partial t} = \frac{\theta''}{G^2} + \frac{6\theta' \cos 3\theta}{hG} - \frac{\theta' G'}{G^3} - \frac{2 \sin 3\theta \cos 3\theta}{h^2}. \tag{6.11}$$

Now (3.13) shows that (6.10) becomes

$$\frac{\partial G}{\partial t} = -\left(9|\nabla \theta|^2 + 3, \left|\frac{\sin 3\theta}{h}\right|^2\right)G,$$

which is the third part of (6.1). Finally, substituting $\cos 3\theta = \frac{h'}{G}$ in (6.11) and using (3.12) gives

$$\frac{\partial \theta}{\partial t} = \Delta\theta - \frac{\sin 6\theta}{h^2},$$

which is the second part of (6.1). □

As discussed at the end of Section 5.2, long-time existence for these evolution equations would be difficult to determine, and in general one should expect singularity formation in finite time.

6.2. The NK soliton equations

Now we turn to the *soliton* equations in the nearly Kähler case. As before, we can without loss of generality reparametrize the local coordinate r so that $G = 1$. Also as in the Calabi–Yau case, we can assume that $X = \nabla k = k' \frac{\partial}{\partial r}$ is a gradient vector field for some function $k = k(r)$ on L^1. We recall again that the soliton equation, as derived in (4.4), is

$$-\Delta_d \psi = \mathcal{L}_{\nabla k}\psi + \lambda\psi = d(\nabla k \lrcorner \psi) + \lambda\psi \tag{6.12}$$

since $d\psi = 0$.

Theorem 6.2. *The coclosed* G_2*-structures which satisfy the soliton equation* (6.12) *when* N^6 *is nearly Kähler are given by* (3.8) *where* $G = 1$ *and the functions* h, θ, *and* k' *satisfy*

$$h' = \cos 3\theta, \tag{6.13}$$

$$0 = \left(h^3 \sin 3\theta\right)'' - 12h \sin 3\theta - \lambda h^3 \sin 3\theta - \left(k' h^3 \sin 3\theta\right)', \tag{6.14}$$

$$0 = \left(h^3 \cos 3\theta\right)' - 3h^2 - \frac{\lambda}{4}h^4 - k' h^3 \cos 3\theta. \tag{6.15}$$

Proof. As in the proof of Theorem 5.2, but this time using (3.14), we find that

$$\begin{aligned} d(\nabla k \lrcorner \psi) &= d\left(k' \frac{\partial}{\partial r} \lrcorner \psi\right) = d\left(\frac{iF^3k'}{2}\Omega - \frac{i\bar{F}^3k'}{2}\bar{\Omega}\right) \\ &= \frac{i}{2}\left(F^3k'\right)' dr \wedge \Omega - \frac{i}{2}\left(\bar{F}^3k'\right)' dr \wedge \bar{\Omega} + \frac{iF^3k'}{2} d\Omega - \frac{i\bar{F}^3k'}{2} d\bar{\Omega} \\ &= \frac{i}{2}\left(F^3k'\right)' dr \wedge \Omega - \frac{i}{2}\left(\bar{F}^3k'\right)' dr \wedge \bar{\Omega} - 2\left(F^3 + \bar{F}^3\right)k' \frac{\omega^2}{2}. \end{aligned}$$

We substitute the above expression into (6.12) and use $G = 1$ and Eqs. (4.7) and (4.6). When we compare coefficients, we find that

$$\left(\frac{i(F^3)'}{2} - \frac{3i}{2}h^2\right)' + 6h\sin 3\theta = \lambda\frac{iF^3}{2} + \frac{i}{2}\left(F^3k'\right)', \qquad -4\left(h^3\cos 3\theta\right)' + 12h^2 = -\lambda h^4 - 2\left(F^3 + \bar{F}^3\right)k'.$$

Taking real and imaginary parts of the first equation, and simplifying all three equations, we get

$$\left(\left(h^3\cos 3\theta\right) - 3h^2\right)' - \lambda h^3\cos 3\theta - \left(k'h^3\cos 3\theta\right)' = 0, \tag{6.16}$$

$$\left(h^3\sin 3\theta\right)'' - 12h\sin 3\theta - \lambda h^3\sin 3\theta - \left(k'h^3\sin 3\theta\right)' = 0, \tag{6.17}$$

$$\left(h^3\cos 3\theta\right)' - 3h^2 - \frac{\lambda}{4}h^4 - k'h^3\cos 3\theta = 0. \tag{6.18}$$

As in Theorem 6.1, this appears to be overdetermined because we also have the $\tau_1 = 0$ assumption (4.5) which is now $\cos 3\theta = h'$, but it is easy to see that with this condition, Eq. (6.16) is a consequence of Eq. (6.18). This completes the proof. □

We now attempt to solve the system of equations in Theorem 6.2. It is easy to spot some particular solutions. For example, if we assume $3\theta = 0$, then (6.14) is trivially satisfied and (6.13) implies that $h = r + b$ for some constant b. Then (6.15) becomes $\lambda(r + b) + 4k' = 0$, which shows that we can find a k' for any choice of λ. Thus one family of solutions is:

$$3\theta = 0, \qquad h = r + b, \qquad k' = -\frac{\lambda}{4}(r + b), \quad \lambda, b \in \mathbb{R}.$$

Similarly, if we assume $3\theta = \pi$, then we find the following family of solutions:

$$3\theta = \pi, \qquad h = -r + b, \qquad k' = \frac{\lambda}{4}(-r + b), \quad \lambda, b \in \mathbb{R}.$$

Since we must have $h > 0$ always, we see that the above two families of solutions are only defined on some proper subinterval of $L^1 = \mathbb{R}^1$. In particular, these families include the case of the Riemannian cone over N^6, given by $h(r) = r$ with $L^1 = (0, \infty)$. The G_2-structure φ is torsion-free in this case and M^7 has G_2 holonomy. This example is entirely analogous to the exhibition of the standard Euclidean metric on $\mathbb{R}^n$ as a nontrivial gradient Ricci soliton.

Another family of special solutions can be found if we assume $3\theta = \pm\frac{\pi}{2}$. In this case (6.13) implies that $h = b$ for some constant $b > 0$, and then (6.15) forces $\lambda = -\frac{12}{b^2}$ and (6.14) then gives $k'' = 0$. Thus another family of solutions is:

$$\theta = \frac{\pi}{2}, \qquad h = b, \qquad k' = c, \qquad \lambda = -\frac{12}{b^2}, \quad b > 0,\ c \in \mathbb{R}.$$

Notice that this family of solutions are all *shrinkers*. In this case the metric (3.4) on M^7 is a Riemannian product.

Finally, we can find a more interesting solution by trying $h(r) = \sin(r)$. The motivation for such an ansatz is that "sine-cone" metrics $g_M = dr^2 + \sin^2(r)g_N$ arise often in the study of Einstein manifolds (see for example [2] or [16]) and the fact that $h' = \cos(3\theta)$. One can check that this ansatz does indeed work and we obtain the following solution:

$$3\theta = r, \qquad h = \sin(r), \qquad k' = 0, \qquad \lambda = -16.$$

This is another shrinking soliton. In this case, $L^1 = (0, \pi)$ and the manifold $M^7 = (0, \pi) \times N^6$ can be compactified to a compact topological space with two "conical singularities." One can also check (for example using the formulae on p. 192 of [27]) that in this case, the metric g_M on M is Einstein. This G_2-structure is *not* torsion-free, but by Eq. (6.12) the 3-form φ is an eigenform (with eigenvalue 16) of its induced Hodge Laplacian Δ_d.

In the general case, we can reduce the equations of Theorem 6.2 to a single *third order* nonlinear ordinary differential equation for h as follows. Let us assume that $h' = \cos 3\theta$ is never zero. We know that $h = r + b$ and $\theta = 0$ is a solution with this property, so we are looking for other solutions close to this one. First, we substitute (6.13) into (6.15) to obtain

$$
\begin{aligned}
0 &= \left(h^3 h'\right)' - 3h^2 - \frac{\lambda}{4} h^4 - k' h^3 h' \\
&= 3h^2 \left(h'\right)^2 + h^3 h'' - 3h^2 - \frac{\lambda}{4} h^4 - h^3 h' k'.
\end{aligned}
$$

We can solve the above expression for $h^3 k'$ as:

$$
h^3 k' = 3h^2 h' + \frac{h^3 h''}{h'} - \frac{3h^2}{h'} - \frac{\lambda h^4}{4h'}. \tag{6.19}
$$

We will also need the derivative of the above expression:

$$
\begin{aligned}
\left(h^3 k'\right)' &= \left(6h\left(h'\right)^2 + 3h^2 h''\right) + \left(3h^2 h'' + \frac{h^3 h'''}{h'} - \frac{h^3 (h'')^2}{(h')^2}\right) + \left(-6h + \frac{3h^2 h''}{(h')^2}\right) + \left(-\lambda h^3 + \frac{\lambda h^4 h''}{4(h')^2}\right) \\
&= 6h\left(h'\right)^2 + 6h^2 h'' - 6h - \lambda h^3 + \frac{h^3 h'''}{h'} + \frac{3h^2 h''}{(h')^2} + \frac{\lambda h^4 h''}{4(h')^2} - \frac{h^3 (h'')^2}{(h')^2}. \qquad (6.20)
\end{aligned}
$$

Let us write $u = \sin 3\theta$ to simplify notation. Then Eq. (6.14) is

$$
0 = \left(h^3 u\right)'' - 12hu - \lambda h^3 u - \left(h^3 k' u\right)' \tag{6.21}
$$

$$
= 6h\left(h'\right)^2 u + 3h^2 h'' u + 6h^2 h' u' + h^3 u'' - 12hu - \lambda h^3 u - \left(h^3 k'\right)' u - \left(h^3 k'\right) u'. \tag{6.22}
$$

We can substitute (6.19) and (6.20) into (6.22) to completely eliminate k'. After some simplification, the end result is

$$
\begin{aligned}
0 = {} & u''\left(h^3\right) + u'\left(3h^2 h' - \frac{h^3 h''}{h'} + \frac{3h^2}{h'} + \frac{\lambda h^4}{4h'}\right) \\
& + u\left(-3h^2 h'' - 6h - \frac{h^3 h'''}{h'} - \frac{3h^2 h''}{(h')^2} - \frac{\lambda h^4 h''}{4(h')^2} + \frac{h^3 (h'')^2}{(h')^2}\right). \qquad (6.23)
\end{aligned}
$$

The next step is to eliminate $u = \sin 3\theta$ from the above equation. Since $h' = \cos 3\theta$, we have

$$
u^2 = 1 - \left(h'\right)^2. \tag{6.24}
$$

We can differentiate the above equation to get

$$
uu' = -h' h''. \tag{6.25}
$$

Now we differentiate (6.25), multiply both sides by u^2, and use both (6.24) and (6.25) again:

$$
\begin{aligned}
&\left(u'\right)^2 + uu'' = -\left(\left(h''\right)^2 + h' h'''\right), \\
&u^2\left(\left(u'\right)^2 + uu''\right) = -u^2\left(\left(h''\right)^2 + h' h'''\right), \\
&\left(uu'\right)^2 + u^3 u'' = -\left(1 - \left(h'\right)^2\right)\left(\left(h''\right)^2 + h' h'''\right), \\
&\left(-h' h''\right)^2 + u^3 u'' = -\left(h''\right)^2 - h' h''' + \left(h'\right)^2 \left(h''\right)^2 + \left(h'\right)^3 h'''.
\end{aligned}
$$

From the above we find

$$
u^3 u'' = \left(h'\right)^3 h''' - h' h''' - \left(h''\right)^2. \tag{6.26}
$$

We can now multiply Eq. (6.23) by u^3 and substitute (6.24), (6.25), and (6.26) for $u^4 = (u^2)^2$, $u^3 u' = u^2(uu')$, and $u^3 u'' = u^2(uu'')$. We can then multiply through by $(h')^2$ to clear the denominators. This eliminates u completely and leaves only a third order nonlinear (polynomial) ordinary differential equation for h. The result is:

$$
\begin{aligned}
& h^3\left(h'\right)^3 h''' - h^3 h' h''' - 2h^3\left(h'\right)^2\left(h''\right)^2 + 3h^2\left(h'\right)^4 h'' - 6h\left(h'\right)^2 + h^3\left(h''\right)^2 - 3h^2 h'' \\
& \quad + 12h\left(h'\right)^4 - 6h\left(h'\right)^6 + \frac{\lambda}{4} h^4\left(h'\right)^2 h'' - \frac{\lambda}{4} h^4 h'' = 0. \qquad (6.27)
\end{aligned}
$$

If one can solve this equation, then we also get the solution *algebraically* for $u = \sin 3\theta$ from (6.24) and for k' from (6.19). However, there does not appear to be an integrating factor for this differential equation and hence it is not clear if the general solution can be found explicitly, as is often (but not always) the case with cohomogeneity-one solitons for geometric flows. See [14] for examples of cohomogeneity-one Ricci solitons which were not exactly integrable, but where a dynamical systems analysis was possible.

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