Mean Curvature Flows and Isotopy of Maps Between Spheres

MAO-PEI TSUI
AND
MU-TAO WANG
Columbia University

Abstract

Let f be a smooth map between unit spheres of possibly different dimensions. We prove the global existence and convergence of the mean curvature flow of the graph of f under various conditions. A corollary is that any area-decreasing map between unit spheres (of possibly different dimensions) is isotopic to a constant map. © 2004 Wiley Periodicals, Inc.

1 Introduction

Let Σ_1 and Σ_2 be two compact Riemannian manifolds and $M = \Sigma_1 \times \Sigma_2$ be the product manifold. We consider a smooth map $f : \Sigma_1 \to \Sigma_2$ and denote the graph of f by Σ; Σ is a submanifold of M by the embedding $id \times f$. In [13, 14, 15] the second author studied the deformation of f by the mean curvature flow (see also the work of Chen, Li, and Tian [3]). The idea is to deform Σ along the direction of its mean curvature vector in M with the hope that Σ will remain a graph. This is the negative gradient flow of the volume functional, and a stationary point is a "minimal map" introduced by Schoen in [10]. In [15] the second author proved various long-time existence and convergence results of graphical mean curvature flows in arbitrary codimensions under assumptions on the Jacobian of the projection from Σ to Σ_1. This quantity is denoted by $*\Omega$ in [15] and $*\Omega > 0$ if and only if Σ is a graph over Σ_1 by the implicit function theorem. A crucial observation in [15] is that $*\Omega$ is a monotone quantity under the mean curvature flow when $*\Omega > 1/\sqrt{2}$. The case when $\Sigma_1 = \mathbb{R}^n$ and $\Sigma_2 = \mathbb{R}$ corresponds to the mean curvature flow of the graph of $f : \mathbb{R}^n \to \mathbb{R}$ and was studied extensively by Ecker and Huisken in [4, 5]. In this case $*\Omega = 1/(\sqrt{1 + |\nabla f|^2})$ plays an important role in their estimates.

In this paper we discover new positive geometric quantities preserved by the graphical mean curvature flow. To describe these results, we recall that the differential of f, df, at each point of Σ_1 is a linear map between the tangent spaces. The Riemannian structures enables us to define the adjoint of df. Let $\{\lambda_i\}$ denote the eigenvalues of $\sqrt{(df)^{\mathsf{T}} df}$, or the singular values of df, where $(df)^{\mathsf{T}}$ is the

Communications on Pure and Applied Mathematics, Vol. LVII, 1110–1126 (2004)

adjoint of df. Note that λ_i is always nonnegative. We say f is an *area-decreasing map* if $\lambda_i \lambda_j < 1$ for any $i \neq j$ at each point. In particular, f is area decreasing if the df has rank 1 everywhere. Under this condition, the second author proves the Bernstein-type theorem [17] and interior gradient estimates [19] for solutions of the minimal surface system. It is also proven in [18] that the set of graphs of area-decreasing linear transformations forms a convex subset of the Grassmannian. We prove that this condition is preserved along the mean curvature flow and the following global existence and convergence theorem:

THEOREM 1.1 *Let Σ_1 and Σ_2 be compact Riemannian manifolds of constant curvature k_1 and k_2, respectively. Suppose $k_1 \geq |k_2|$, $k_1 + k_2 > 0$, and $\dim(\Sigma_1) \geq 2$. If f is a smooth, area-decreasing map from Σ_1 to Σ_2, the mean curvature flow of the graph of f remains the graph of an area-decreasing map, exists for all time, and converges smoothly to the graph of a constant map.*

We remark that the condition $k_1 \geq |k_2|$ is enough to prove the long-time existence of the flow. The following is an application to determine when a map between spheres is homotopically trivial:

COROLLARY 1.2 *Any area-decreasing map from $\mathbb{S}^n$ to $\mathbb{S}^m$ with $n \geq 2$ is homotopically trivial.*

When $m = 1$, the area-decreasing condition always holds and the above statement follows from the fact that $\pi_n(\mathbb{S}^1)$ is trivial for $n \geq 2$. We remark that the result when $m = 2$ is proven by the second author in [16] using a somewhat different method. The higher homotopy groups $\pi_n(\mathbb{S}^m)$ have been computed in many cases, and it is known that homotopically nontrivial maps do exist when $n \geq m$. Since an area-decreasing map may still be surjective when $n > m$, we do not know any topological method that would imply such a conclusion. The famous work of Eells and Sampson [6] uses the harmonic map heat flow to deform maps between Riemannian manifolds. The flow exists for all time and converges nicely when the curvature of the target space is nonpositive. However, the flow may develop singularities for positively curved target spaces.

2 Preliminaries

In this section, we recall notation and formulae for mean curvature flows. Let $f : \Sigma_1 \to \Sigma_2$ be a smooth map between Riemannian manifolds. The graph of f is an embedded submanifold Σ in $M = \Sigma_1 \times \Sigma_2$. At any point of Σ, the tangent space of M, TM, splits into the direct sum of the tangent space of Σ, $T\Sigma$, and the normal space $N\Sigma$, the orthogonal complement of the tangent space $T\Sigma$ in TM. There are isomorphisms $T\Sigma_1 \to T\Sigma$ by $X \mapsto X + df(X)$ and $T\Sigma_2 \to N\Sigma$ by $Y \mapsto Y - (df)^\mathsf{T}(Y)$ where $(df)^\mathsf{T} : T\Sigma_2 \to T\Sigma_1$ is the adjoint of df.

We assume the mean curvature flow of Σ can be written as a graph of f_t for $t \in [0, \epsilon)$ and derive the equation satisfied by f_t. The mean curvature flow is given by a smooth family of immersions F_t of Σ into M that satisfies

$$\left(\frac{\partial F}{\partial t}\right)^{\perp} = H$$

where H is the mean curvature vector in M and $(\,\cdot\,)^{\perp}$ denotes the projection onto the normal space $N\Sigma$. Notice that we do not require $\frac{\partial F}{\partial t}$ to be in the normal direction since the difference is only a tangential diffeomorphism (see, e.g., White [20] for the issue of parametrization). By the definition of the mean curvature vector, this equation is equivalent to

$$\left(\frac{\partial F}{\partial t}\right)^{\perp} = \left(\Lambda^{ij}\nabla^M_{\partial F/\partial x^i}\frac{\partial F}{\partial x^j}\right)^{\perp}$$

where Λ^{ij} is the inverse to the induced metric $\Lambda_{ij} = \langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\rangle$ on Σ.

In terms of coordinates $\{y^A\}_{A=1,\dots,n+m}$ on M, we have

$$\Lambda^{ij}\nabla^M_{\partial F/\partial x^j}\frac{\partial F}{\partial x^i} = \Lambda^{ij}\left(\frac{\partial^2 F^A}{\partial x^i \partial x^j} + \frac{\partial F^B}{\partial x^i}\frac{\partial F^C}{\partial x^j}\Gamma^A_{BC}\right)\frac{\partial}{\partial y^A}$$

where Γ^A_{BC} is the Christoffel symbol of M and thus

$$\left(\Lambda^{ij}\nabla^M_{\partial F/\partial x^j}\frac{\partial F}{\partial x^i}\right)^{\perp} = \Lambda^{ij}\left(\frac{\partial^2 F^A}{\partial x^i \partial x^j} + \frac{\partial F^B}{\partial x^i}\frac{\partial F^C}{\partial x^j}\Gamma^A_{BC} - \tilde{\Gamma}^k_{ij}\frac{\partial F^A}{\partial x^k}\right)\frac{\partial}{\partial y^A}$$

where $\tilde{\Gamma}^k_{ij}$ is the Christoffel symbol of the induced metric on Σ.

By assumption, the embedding is given by the graph of f_t. We fix a coordinate system $\{x^i\}$ on Σ_1 and consider $F : \Sigma_1 \times [0, T) \to M$ given by

$$F(x^1, \dots, x^n, t) = (x^1, \dots, x^n, f^{n+1}, \dots, f^{n+m})\,.$$

We shall use $i, j, k, l, \dots = 1, \dots, n$ and $\alpha, \beta, \gamma = n+1, \dots, n+m$ for the indices. Of course, $f^\alpha = f^\alpha(x^1, \dots, x^n, t)$ is time dependent.

Therefore $\frac{\partial F}{\partial t} = \frac{\partial f^\alpha}{\partial t}\frac{\partial}{\partial y^\alpha}$ and

$$\Lambda^{ij}\left(\frac{\partial^2 F^A}{\partial x^i \partial x^j} + \frac{\partial F^B}{\partial x^i}\frac{\partial F^C}{\partial x^j}\Gamma^A_{BC}\right)\frac{\partial}{\partial y^A} =$$
$$\Lambda^{ij}\left(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j}\frac{\partial}{\partial y^\alpha} + \Gamma^l_{ij}\frac{\partial}{\partial y^l} + \frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\Gamma^\alpha_{\beta\gamma}\frac{\partial}{\partial y^\alpha}\right).$$

Thus the mean curvature flow equation is equivalent to the normal part of

$$\left[\frac{\partial f^\alpha}{\partial t} - \Lambda^{ij}\left(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} + \frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\Gamma^\alpha_{\beta\gamma}\right)\right]\frac{\partial}{\partial y^\alpha} - \Lambda^{ij}\Gamma^l_{ij}\frac{\partial}{\partial y^l}$$

being 0.

Given any vector $a^i\frac{\partial}{\partial y^i} + b^\alpha\frac{\partial}{\partial y^\alpha}$, the equation with the normal part being 0 is equivalent to

$$b^\alpha - a^i\frac{\partial f^\alpha}{\partial x^i} = 0 \tag{2.1}$$

for each α. Therefore we obtain the evolution equation for f

$$\frac{\partial f^\alpha}{\partial t} - \Lambda^{ij}\left(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} + \frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\Gamma^\alpha_{\beta\gamma} + \Gamma^k_{ij}\frac{\partial f^\alpha}{\partial x^k}\right) = 0\,. \tag{2.2}$$

where Λ^{ij} is the inverse to $g_{ij} + h_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}$ and

$$g_{ij} = \left\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right\rangle \quad \text{and} \quad h_{\alpha\beta} = \left\langle \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right\rangle$$

are the Riemannian metrics on Σ_1 and Σ_2, respectively. Γ^k_{ij} and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of g_{ij} and $h_{\alpha\beta}$, respectively.

(2.2) is a nonlinear parabolic system and the usual derivative estimates do not apply to these equations. However, the second author in [15] identifies a geometric quantity in terms of the derivatives of f^α that satisfies the maximum principle; this quantity and its evolution equation are recalled in the next section.

3 Two Evolution Equations

In this section, we recall two evolution equations along the mean curvature flow. The basic setup is a mean curvature flow $F : \Sigma \times [0, T) \to M$ of an n dimensional submanifold Σ inside an $(n + m)$–dimensional Riemannian manifold M. Given any parallel tensor on M, we may consider the pullback tensor by F_t and consider the evolution equation with respect to the time-dependent induced metric on $F_t(\Sigma) = \Sigma_t$. For the purpose of applying the maximum principle, it suffices to derive the equation at a space-time point. We write all geometric quantities in terms of orthonormal frames keeping in mind that all quantities are defined independently of choice of frame. At any point $p \in \Sigma_t$, we choose any orthonormal frame $\{e_i\}_{i=1,\dots,n}$ for $T_p\Sigma_t$ and $\{e_\alpha\}_{\alpha=n+1,\dots,n+m}$ for $N_p\Sigma_t$. The second fundamental form $h_{\alpha ij}$ is denoted by $h_{\alpha ij} = \langle \nabla^M_{e_i} e_j, e_\alpha\rangle$, and the mean curvature vector is denoted by $H_\alpha = \sum_i h_{\alpha ii}$. For any j, k, we pretend

$$h_{n+i,jk} = 0$$

if $i > m$.

When $M = \Sigma_1 \times \Sigma_2$ is the product of Σ_1 and Σ_2, we denote the projections by $\pi_1 : M \to \Sigma_1$ and $\pi_2 : M \to \Sigma_2$. By abusing notation, we also denote the differentials by $\pi_1 : T_pM \to T_{\pi_1(p)}\Sigma_1$ and $\pi_1 : T_pM \to T_{\pi_2(p)}\Sigma_2$ at any point $p \in M$. The volume form Ω of Σ_1 can be extended to a parallel n-form on M. For an oriented orthonormal basis $e_1, \dots, e_n$ of $T_p\Sigma$, $\Omega(e_1, \dots, e_n) = \Omega(\pi_1(e_1), \dots, \pi_1(e_n))$ is the Jacobian of the projection from $T_p\Sigma$ to $T_{\pi_1(p)}\Sigma_1$. This can also be considered as the pairing between the n-form Ω and the n-vector $e_1 \wedge \cdots \wedge e_n$ representing $T_p\Sigma$. We use $*\Omega$ to denote this function as p varies along Σ. By the implicit function theorem, $*\Omega > 0$ at p if and only if Σ is locally a graph over Σ_1 at p. The evolution equation of $*\Omega$ is calculated in [15, prop. 3.2].

When Σ is the graph of $f : \Sigma_1 \to \Sigma_2$, the equation at each point can be written in terms of singular values of df and special bases adapted to df. Denote the singular values of df, or eigenvalues of $\sqrt{(df)^{\mathsf{T}} df}$, by $\{\lambda_i\}_{i=1,\ldots,n}$. Let r denote the rank of df. We can rearrange them so that $\lambda_i = 0$ when $i > r$. By singular value decomposition, there exist orthonormal bases $\{a_i\}_{i=1,\ldots,n}$ for $T_{\pi_1(p)}\Sigma_1$ and $\{a_\alpha\}_{\alpha=n+1,\ldots,n+m}$ for $T_{\pi_2(p)}\Sigma_2$ such that

$$df(a_i) = \lambda_i a_{n+i}$$

for $i \le r$ and $df(a_i) = 0$ for $i > r$. Moreover,

$$e_i = \begin{cases} \frac{1}{\sqrt{1+\lambda_i^2}}(a_i + \lambda_i a_{n+i}) & \text{if } 1 \le i \le r \\ a_i & \text{if } r+1 \le i \le n \end{cases} \tag{3.1}$$

becomes an orthonormal basis for $T_p\Sigma$ and

$$e_{n+p} = \begin{cases} \frac{1}{\sqrt{1+\lambda_p^2}}(a_{n+p} - \lambda_p a_p) & \text{if } 1 \le p \le r \\ a_{n+p} & \text{if } r+1 \le p \le m \end{cases} \tag{3.2}$$

becomes an orthonormal basis for $N_p\Sigma$.

In terms of the singular values λ_i,

$$*\Omega = \frac{1}{\sqrt{\prod_{i=1}^n (1+\lambda_i^2)}}. \tag{3.3}$$

With all the notation understood, the following result is essentially derived in [15, prop. 3.2] by noting that $(\ln *\Omega)_k = -(\sum_i \lambda_i h_{n+i,ik})$.

PROPOSITION 3.1 *Suppose $M = \Sigma_1 \times \Sigma_2$ and Σ_1 and Σ_2 are compact Riemannian manifolds of constant curvature k_1 and k_2, respectively. With respect to the particular bases given by the singular value decomposition of df,* $\ln *\Omega$ *satisfies the following equation*:

$$\begin{aligned} \left(\frac{d}{dt} - \Delta\right) \ln *\Omega = &\sum_{\alpha,i,k} h_{\alpha ik}^2 + \sum_{k,i} \lambda_i^2 h_{n+i,ik}^2 + 2\sum_{k,i<j} \lambda_i\lambda_j h_{n+j,ik} h_{n+i,jk} \\ &+ \sum_i \frac{\lambda_i^2}{1+\lambda_i^2}\Big[(k_1+k_2)\Big(\sum_{j\ne i}\frac{1}{1+\lambda_j^2}\Big) + k_2(1-n)\Big]. \end{aligned} \tag{3.4}$$

Next we recall the evolution equation of parallel 2-tensors from [12]. The calculation indeed already appears in [14]. The equation will be used later to obtain more refined information. Given a parallel 2-tensor S on M, we consider the evolution of S restricted to Σ_t. This is a family of time-dependent symmetric 2-tensors on Σ_t.

PROPOSITION 3.2 *Let S be a parallel* 2*-tensor on M. Then the pullback of S to Σ_t satisfies the following equation*:

$$\begin{aligned} \left(\frac{d}{dt} - \Delta\right) S_{ij} = &-h_{\alpha il} H_\alpha S_{lj} - h_{\alpha jl} H_\alpha S_{li} + R_{kik\alpha} S_{\alpha j} + R_{kjk\alpha} S_{\alpha i} \\ &+ h_{\alpha kl} h_{\alpha ki} S_{lj} + h_{\alpha kl} h_{\alpha kj} S_{li} - 2h_{\alpha ki} h_{\beta kj} S_{\alpha\beta} \end{aligned} \tag{3.5}$$

where Δ is the rough Laplacian on 2-tensors over Σ_t and $S_{\alpha i} = S(e_\alpha, e_i)$, $S_{\alpha\beta} = S(e_\alpha, e_\beta)$, and $R_{kik\alpha} = R(e_k, e_i, e_k, e_\alpha)$ is the curvature of M.

The evolution equations (3.5) of S can be written in terms of evolving orthonormal frames as in Hamilton [8] if the orthonormal frames

$$F = \{F_1, \ldots, F_a, \ldots, F_n\} \tag{3.6}$$

are given in local coordinates by

$$F_a = F_a^i \frac{\partial}{\partial x_i}\,.$$

To keep them orthonormal, i.e,. $g_{ij} F_a^i F_b^j = \delta_{ab}$, we evolve F by the formula

$$\frac{\partial}{\partial t} F_a^i = g^{ij} g^{\alpha\beta} h_{\alpha jl} H_\beta F_a^l\,.$$

Let $S_{ab} = S_{ij} F_a^i F_b^j$ be the components of S in F. Then S_{ab} satisfies the following equation:

$$\begin{aligned} \left(\frac{d}{dt} - \Delta\right) S_{ab} = {} & R_{ca c\alpha} S_{\alpha b} + R_{cbc\alpha} S_{\alpha a} + h_{\alpha cd} h_{\alpha ca} S_{db} \\ & + h_{\alpha cd} h_{\alpha cb} S_{da} - 2 h_{\alpha ca} h_{\beta cb} S_{\alpha\beta}\,. \end{aligned} \tag{3.7}$$

4 Preserving the Distance-Decreasing Condition

In this section we show that the *distance-decreasing condition* $|df| < 1$, or each singular value $\lambda_i < 1$, is preserved by the mean curvature flow. This result will not be used in the proof of Theorem 1.1. But the proof of Theorem 1.1 depends on the computation in this section. The tangent space of $M = \Sigma_1 \times \Sigma_2$ is identified with $T\Sigma_1 \oplus T\Sigma_2$. Let π_1 and π_2 denote the projection onto the first and second summand in the splitting. We define the parallel symmetric 2-tensor S by

$$S(X, Y) = \langle \pi_1(X), \pi_1(Y)\rangle - \langle \pi_2(X), \pi_2(Y)\rangle \tag{4.1}$$

for any $X, Y \in TM$.

Let Σ be the graph of $f : \Sigma_1 \to \Sigma_1 \times \Sigma_2$. S restricts to a symmetric 2-tensor on Σ, and we can represent S in terms of the orthonormal basis (3.1).

Let r denote the rank of df. By (3.1), it is not hard to check

$$\begin{aligned} &\pi_1(e_i) = \frac{a_i}{\sqrt{1+\lambda_i{}^2}}\,, && \pi_2(e_i) = \frac{\lambda_i a_{n+i}}{\sqrt{1+\lambda_i{}^2}}\,, && \text{for } 1 \le i \le r, \\ &\pi_1(e_i) = a_i\,, && \pi_2(e_i) = 0\,, && \text{for } r+1 \le i \le n. \end{aligned} \tag{4.2}$$

Similarly, by (3.2) we have

$$\begin{aligned} &\pi_1(e_{n+p}) = \frac{-\lambda_p a_p}{\sqrt{1+\lambda_p{}^2}}\,, && \pi_2(e_{n+p}) = \frac{a_{n+p}}{\sqrt{1+\lambda_p{}^2}}\,, && \text{for } 1 \le p \le r, \\ &\pi_1(e_{n+p}) = 0\,, && \pi_2(e_{n+p}) = a_{n+p}\,, && \text{for } r+1 \le p \le m. \end{aligned} \tag{4.3}$$

From the definition of S, we have

$$S(e_i, e_j) = \frac{1-\lambda_i^2}{1+{\lambda_i}^2}\delta_{ij}\,. \tag{4.4}$$

In particular, the eigenvalues of S are

$$\frac{1-{\lambda_i}^2}{1+{\lambda_i}^2}\,, \quad i = 1, \dots, n. \tag{4.5}$$

Notice that S is positive definite if and only if

$$\lambda_i < 1$$

for any singular value λ_i of df.

Now, at each point we express S in terms of the orthonormal basis $\{e_i\}_{i=1,\dots,n}$ and $\{e_\alpha\}_{\alpha=n+1,\dots,n+m}$. Let $I_{k\times k}$ denote a $k\times k$ identity matrix. Then S can be written in the block form

$$S = (S(e_k, e_l))_{1\le k,l\le n+m} = \begin{pmatrix} B & 0 & D & 0 \\ 0 & I_{n-r\times n-r} & 0 & 0 \\ D & 0 & -B & 0 \\ 0 & 0 & 0 & -I_{m-r\times m-r} \end{pmatrix} \tag{4.6}$$

where B and D are $r \times r$ matrices with

$$B_{ij} = S(e_i, e_j) = \frac{1-\lambda_i^2}{1+\lambda_i^2}\delta_{ij} \quad \text{and} \quad D_{ij} = S(e_i, e_{n+j}) = \frac{-2\lambda_i}{1+\lambda_i^2}\delta_{ij}$$

for $1 \le i, j \le r$. We show that the positivity of S is preserved by the mean curvature flow. We remark that a similar positive definite tensor has been considered for the Lagrangian mean curvature flow in Smoczyk [11] and Smoczyk and Wang [12]. The following lemma shows that the distance-decreasing condition is preserved by the mean curvature flow if $k_1 \ge |k_2|$.

LEMMA 4.1 *The condition*

$$T_{ij} = S_{ij} - \epsilon g_{ij} > 0 \quad \textit{for some } \epsilon \ge 0 \tag{4.7}$$

is preserved by the mean curvature flow if $k_1 \ge |k_2|$.

PROOF: We compute the evolution equation for T_{ij}. From Proposition 3.2 and

$$\frac{\partial}{\partial t} g_{ij} = -2h_{\alpha ij} H_\alpha\,,$$

we have

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) T_{ij} = &-h_{\alpha il} H_\alpha T_{lj} - h_{\alpha jl} H_\alpha T_{li} + R_{kik\alpha} S_{\alpha j} + R_{kjk\alpha} S_{\alpha i} \\ &+ h_{\alpha kl} h_{\alpha ki} T_{lj} + h_{\alpha kl} h_{\alpha kj} T_{li} + 2\epsilon h_{\alpha ki} h_{\alpha kj} \\ &- 2h_{\alpha ki} h_{\beta kj} S_{\alpha\beta}\,.\end{aligned} \tag{4.8}$$

To apply Hamilton's maximum principle, it suffices to prove that $N_{ij} V^i V^j \ge 0$ for any null eigenvector V of T_{ij}, where N_{ij} is the right-hand side of (4.8). Since V

is a null eigenvector of T_{ij}, it satisfies $\sum_j T_{ij} V^j = 0$ for any i, and thus $N_{ij} V^i V^j$ is equal to

$$2\epsilon h_{\alpha ki} h_{\alpha kj} V^i V^j + 2R_{kik\alpha} S_{\alpha j} V^i V^j - 2h_{\alpha ki} h_{\beta kj} S_{\alpha\beta} V^i V^j . \tag{4.9}$$

Obviously, the first term of (4.9) is nonnegative. Applying the relation in (4.6) to the last term of (4.9), we obtain

$$\begin{aligned} -2h_{\alpha ki} h_{\beta kj} S_{\alpha\beta} V^i V^j = & \sum_{1 \le p,q \le r} 2h_{n+pki} h_{n+qkj} S_{pq} V^i V^j \\ & + \sum_{r+1 \le p,q \le m} 2h_{n+pki} h_{n+qkj} V^i V^j . \end{aligned}$$

Since $T_{pq} \ge 0$ implies that $S_{pq} \ge \epsilon g_{pq}$, we obtain $-2h_{\alpha ki} h_{\beta kj} S_{\alpha\beta} V^i V^j \ge 0$. In the next lemma we show that $R_{kik\alpha} S_{\alpha j}$ is nonnegative definite whenever S_{ij} is under the curvature assumption $k_1 \ge |k_2|$. □

LEMMA 4.2

$$R_{kik\alpha} S_{\alpha j} = \frac{\lambda_i^2}{(1+\lambda_i^2)^2} \left[(k_1 - k_2)(n-1) + (k_1 + k_2) \sum_{k \ne i} \frac{1 - \lambda_k^2}{1 + \lambda_k^2} \right] \delta_{ij} . \tag{4.10}$$

PROOF: We follow the calculation of the curvature terms in [15]:

$$\begin{aligned} & \sum_k R(e_\alpha, e_k, e_k, e_i) \\ & \quad = \sum_k R_1(\pi_1(e_\alpha), \pi_1(e_k), \pi_1(e_k), \pi_1(e_i)) + R_2(\pi_2(e_\alpha), \pi_2(e_k), \pi_2(e_k), \pi_2(e_i)) \\ & \quad = \sum_k k_1 [\langle \pi_1(e_\alpha), \pi_1(e_k) \rangle \langle \pi_1(e_k), \pi_1(e_i) \rangle - \langle \pi_1(e_\alpha), \pi_1(e_i) \rangle \langle \pi_1(e_k), \pi_1(e_k) \rangle] \\ & \qquad + k_2 [\langle \pi_2(e_\alpha), \pi_2(e_k) \rangle \langle \pi_2(e_k), \pi_2(e_i) \rangle - \langle \pi_2(e_\alpha), \pi_2(e_i) \rangle \langle \pi_2(e_k), \pi_2(e_k) \rangle] . \end{aligned}$$

Notice that $\langle \pi_2(X), \pi_2(Y) \rangle = \langle X, Y \rangle - \langle \pi_1(X), \pi_1(Y) \rangle$ since $T\Sigma_1 \perp T\Sigma_2$. Therefore

$$\begin{aligned} & \sum_k R(e_\alpha, e_k, e_k, e_i) \\ & \quad = \sum_k (k_1 + k_2) \big[\langle \pi_1(e_\alpha), \pi_1(e_k) \rangle \langle \pi_1(e_k), \pi_1(e_i) \rangle - \langle \pi_1(e_\alpha), \pi_1(e_i) \rangle |\pi_1(e_k)|^2 \big] \\ & \qquad + k_2 (n-1) \langle \pi_1(e_\alpha), \pi_1(e_i) \rangle . \end{aligned}$$

Now using this equation,

$$\pi_1(e_\alpha) = -\lambda_p \pi_1(e_p) \delta_{\alpha, n+p} \quad \text{and} \quad S(e_j, e_{n+p}) = -\frac{2\lambda_j \delta_{jp}}{1 + \lambda_j{}^2}$$

in (4.6), we can express

$$\sum_{\alpha,k} R_{kik\alpha} S_{\alpha j} = -\sum_{p,k} R_{n+p,kki} S_{n+p,j}$$

as

$$-\frac{2\lambda_i^2}{1+\lambda_i{}^2}\left\{(k_1+k_2)\left[\frac{\delta_{ij}}{(1+\lambda_i{}^2)^2} - \frac{\delta_{ij}}{1+\lambda_i^2}\sum_k |\pi_1(e_k)|^2\right] + k_2(n-1)\frac{\delta_{ij}}{1+\lambda_i{}^2}\right\}.$$

Recalling that $|\pi_1(e_k)|^2 = 1/(1+\lambda_k^2)$, we obtain

$$R_{kik\alpha} S_{\alpha j} = \frac{2\lambda_i^2 \delta_{ij}}{(1+\lambda_i^2)^2}\left[(k_1+k_2)\left(\sum_{k\neq i}\frac{1}{1+\lambda_k^2}\right) + k_2(1-n)\right].$$

This can be further simplified by noting

$$(k_1+k_2)\left(\sum_{k\neq i}\frac{1}{1+\lambda_k^2}\right) + k_2(1-n) = \frac{(k_1-k_2)(n-1)}{2} + (k_1+k_2)\sum_{k\neq i}\frac{1-\lambda_k^2}{2(1+\lambda_k^2)} \tag{4.11}$$

where we use the following identity for each i:

$$\left(\sum_{k\neq i}\frac{1}{1+\lambda_k^2}\right) - \frac{n-1}{2} = \sum_{k\neq i}\left(\frac{1}{1+\lambda_k^2} - \frac{1}{2}\right) = \sum_{k\neq i}\frac{1-\lambda_k^2}{2(1+\lambda_k^2)}.$$

□

5 Preserving the Area-Decreasing Condition

In this section we show that the area-decreasing condition is preserved along the mean curvature flow. In the following, we require that $n = \dim(\Sigma_1) \geq 2$. By (4.5), the sum of any two eigenvalues of S is

$$\frac{1-\lambda_i^2}{1+\lambda_i^2} + \frac{1-\lambda_j^2}{1+\lambda_j^2} = \frac{2(1-\lambda_i^2\lambda_j^2)}{(1+\lambda_i^2)(1+\lambda_j^2)}. \tag{5.1}$$

Therefore the area-decreasing condition $\lambda_i\lambda_j < 1$ for $i \neq j$ is equivalent to the two-positivity of S, i.e., the sum of any two eigenvalues is positive. We remark that the curvature operator being two-positive is preserved by the Ricci flow; see Chen [2] or Hamilton [8] for details.

The two-positivity of a symmetric 2-tensor P can be related to the convexity of another tensor $P^{[2]}$ associated with P. The following notation is adopted from Caffarelli, Nirenberg, and Spruck [1]. Let P be a self-adjoint operator on an n-dimensional inner product space. From P we can construct a new self-adjoint operator

$$P^{[k]} = \sum_{i=1}^{k} 1 \otimes \cdots \otimes \underset{i}{P} \otimes \cdots \otimes 1$$

acting on the exterior powers Λ^k by

$$P^{[k]}(\omega_1 \wedge \cdots \wedge \omega_k) = \sum_{i=1}^{k} \omega_1 \wedge \cdots \wedge P(\omega_i) \wedge \cdots \wedge \omega_k \,.$$

With the definition of $P^{[k]}$, we have the following lemma:

LEMMA 5.1 *Let $\mu_1 \le \mu_2 \le \cdots \le \mu_n$ be the eigenvalues of P with corresponding eigenvectors $v_1, \dots, v_n$. Then $P^{[k]}$ has eigenvalues $\mu_{i_1} + \cdots + \mu_{i_k}$ and eigenvectors $v_{i_1} \wedge \cdots \wedge v_{i_k}$, $i_1 < i_2 \cdots < i_k$.*

Recall that the Riemannian metric g and S are both in $T\Sigma \odot T\Sigma$, the space of symmetric 2-tensors on Σ. We can identify S with a self-adjoint operator on the tangent bundle through the metric g. Therefore $S^{[2]}$ and $g^{[2]}$ are both sections of $(\Lambda^2(T\Sigma))^* \odot \Lambda^2(T\Sigma)$ associated to S and g, respectively. We shall use orthonormal frames in the following calculation; this has the advantage that g is the identity matrix, and we will not distinguish between the lower index and the upper index. With the above interpretation and (5.1), we have the following lemma:

LEMMA 5.2 *The area-decreasing condition is equivalent to the convexity of $S^{[2]}$.*

To show that the area-decreasing condition is preserved, it suffices to prove that the convexity of $S^{[2]}$ is preserved. In fact, we prove the stronger result that the convexity of $S^{[2]} - \epsilon g^{[2]}$ for $\epsilon > 0$ is preserved.

We compute the evolution equation of $S^{[2]} - \epsilon g^{[2]}$ in terms of the evolving orthonormal frames $\{F_a\}_{a=1,\dots,n}$ introduced earlier in (3.6). We will use indices $a, b, \dots,$ to denote components in the evolving frames. Denote $S_{ab} = S(F_a, F_b)$ and $g_{ab} = g(F_a, F_b) = \delta_{ab}$. Since $\{F_a \wedge F_b\}_{a<b}$ form a basis for $\Lambda^2 T\Sigma$, we have

$$\begin{aligned} S^{[2]}(F_a \wedge F_b) &= S(F_a) \wedge F_b + F_a \wedge S(F_b) \\ &= S_{ac} F_c \wedge F_b + F_a \wedge S_{ac} F_c \\ &= \sum_{c<d} (S_{ac}\delta_{bd} + S_{bd}\delta_{ac} - S_{ad}\delta_{bc} - S_{bc}\delta_{ad}) F_c \wedge F_d \,, \\ g^{[2]}(F_a \wedge F_b) &= \sum_{c<d} (2\delta_{ac}\delta_{bd} - 2\delta_{ad}\delta_{bc}) F_c \wedge F_d \,. \end{aligned} \tag{5.2}$$

We denote $S^{[2]}_{(ab)(cd)} = (S_{ac}\delta_{bd} + S_{bd}\delta_{ac} - S_{ad}\delta_{bc} - S_{bc}\delta_{ad})$ and $g^{[2]}_{(ab)(cd)} = 2\delta_{ac}\delta_{bd} - 2\delta_{ad}\delta_{bc}$. Thus the evolution equation of $S^{[2]} - \epsilon g^{[2]}$ in terms of the evolving orthonormal frames is

$$
\begin{aligned}
&\left(\frac{d}{dt} - \Delta\right)(S_{ac}\delta_{bd} + S_{bd}\delta_{ac} - S_{ad}\delta_{bc} - S_{bc}\delta_{ad} - 2\epsilon\delta_{ac}\delta_{bd} + 2\epsilon\delta_{ad}\delta_{bc})\\
&\quad = R_{eae\alpha}S_{\alpha c}\delta_{bd} + R_{ece\alpha}S_{\alpha a}\delta_{bd} + R_{ebe\alpha}S_{\alpha d}\delta_{ac} + R_{ede\alpha}S_{\alpha b}\delta_{ac}\\
&\qquad - R_{eae\alpha}S_{\alpha d}\delta_{bc} - R_{ede\alpha}S_{\alpha a}\delta_{bc} - R_{ebe\alpha}S_{\alpha c}\delta_{ad} - R_{ece\alpha}S_{\alpha b}\delta_{ad}\\
&\qquad + h_{\alpha ef}h_{\alpha ea}S_{fc}\delta_{bd} + h_{\alpha ef}h_{\alpha ec}S_{fa}\delta_{bd}\\
&\qquad + h_{\alpha ef}h_{\alpha eb}S_{fd}\delta_{ac} + h_{\alpha ef}h_{\alpha ed}S_{fb}\delta_{ac}\\
&\qquad - h_{\alpha ef}h_{\alpha ea}S_{fd}\delta_{bc} - h_{\alpha ef}h_{\alpha ed}S_{fa}\delta_{bc}\\
&\qquad - h_{\alpha ef}h_{\alpha eb}S_{fc}\delta_{ad} - h_{\alpha ef}h_{\alpha ec}S_{fb}\delta_{ad}\\
&\qquad - 2h_{\alpha ea}h_{\beta ec}S_{\alpha\beta}\delta_{bd} - 2h_{\alpha eb}h_{\beta ed}S_{\alpha\beta}\delta_{ac}\\
&\qquad + 2h_{\alpha ea}h_{\beta ed}S_{\alpha\beta}\delta_{bc} + 2h_{\alpha eb}h_{\beta ec}S_{\alpha\beta}\delta_{ad}\,.
\end{aligned}
\tag{5.3}
$$

Now, we are ready to prove that the area-decreasing condition is preserved along the mean curvature flow.

LEMMA 5.3 *Under the assumption of Theorem* 1.1, *with S defined in* (4.1) *and* $S^{[2]}$ *defined in* (5.2), *suppose there exists an* $\epsilon > 0$ *such that*

$$S^{[2]} - \epsilon g^{[2]} \geq 0 \tag{5.4}$$

holds on the initial graph. Then this is preserved along the mean curvature flow.

PROOF: Set

$$M_\eta = S^{[2]} - \epsilon g^{[2]} + \eta t g^{[2]}\,.$$

Suppose the mean curvature flow exists on $[0, T)$. Consider any $T_1 < T$; it suffices to prove that $M_\eta > 0$ on $[0, T_1]$ for all $\eta < \epsilon/(2T_1)$. If not, there will be a first time $0 < t_0 \leq T_1$ where $M_\eta = S^{[2]} - \epsilon g^{[2]} + \eta t g^{[2]}$ is nonnegative definite and has a null eigenvector $V = V^{ab}F_a \wedge F_b$ at some point $x_0 \in \Sigma_{t_0}$. We extend V^{ab} to a parallel tensor in a neighborhood of x_0 along a geodesic emanating out of x_0, and define V^{ab} on $[0, T)$ independent of t. Define $f = \sum_{a<b,c<d} V^{ab}M_{\eta(ab)(cd)}V^{cd}$; then by the equations in (5.2),

$$
\begin{aligned}
f = \sum_{a<b,c<d} &(S_{ac}g_{bd} + S_{bd}g_{ac} - S_{ad}g_{bc} - S_{bc}g_{ad}\\
&+ 2(\eta t - \epsilon)(g_{ac}g_{bd} - g_{ad}g_{bc}))V^{ab}V^{cd}\,.
\end{aligned}
$$

At (x_0, t_0), we have $f = 0$, $\nabla f = 0$, and $\left(\frac{d}{dt} - \Delta\right) f \leq 0$ where ∇ denotes the covariant derivative and Δ denotes the Laplacian on Σ_{t_0}.

We may assume that at (x_0, t_0) the orthonormal frames $\{F_a\}$ are given by $\{e_i\}$ in (3.1). In the following, we use the orthonormal basis $\{e_i\}$ to write down the condition $f = 0$ and $\nabla f = 0$ at (x_0, t_0). The basis $\{e_i\}$ diagonalizes S with eigenvalues $\{\lambda_i\}$, and we order $\{\lambda_i\}$ such that

$$\lambda_1^2 \geq \lambda_2^2 \geq \cdots \geq \lambda_n^2$$

and

$$S_{nn} = \frac{1-\lambda_n^2}{1+\lambda_n^2} \geq \cdots \geq S_{22} = \frac{1-\lambda_2^2}{1+\lambda_2^2} \geq S_{11} = \frac{1-\lambda_1^2}{1+\lambda_1^2}\,. \tag{5.5}$$

It follows from Lemma 5.1 that $\{e_i \wedge e_j\}_{i<j}$ are the eigenvectors of M_η. Thus we may assume that

$$V = e_1 \wedge e_2\,. \tag{5.6}$$

At (x_0, t_0), the condition $f = 0$ is the same as

$$S_{11} + S_{22} = 2\epsilon - 2\eta t_0 > 0\,. \tag{5.7}$$

This is equivalent to

$$\frac{2(1-\lambda_1^2\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)} = 2(\epsilon - \eta t_0) > 0\,.$$

Thus

$$\lambda_1\lambda_2 < 1 \quad \text{and} \quad \lambda_i < 1 \ \text{ for } i \geq 3\,. \tag{5.8}$$

Next, we compute the covariant derivative of the restriction of S on Σ:

$$\begin{aligned}(\nabla_{e_i} S)(e_j, e_k) &= e_i(S(e_j, e_k)) - S(\nabla_{e_i} e_j, e_k) - S(e_j, \nabla_{e_i} e_k)\\ &= S(\nabla^M_{e_i} e_j - \nabla_{e_i} e_j, e_k) + S(e_j, \nabla^M_{e_i} e_k - \nabla_{e_i} e_k)\\ &= h_{\alpha ij} S_{\alpha k} + h_{\beta ik} S_{\beta j}\end{aligned}$$

so

$$S_{jk,i} = h_{\alpha ij} S_{\alpha k} + h_{\beta ik} S_{\beta j}\,.$$

Recall that V_{ab} is parallel at (x_0, t_0) , $V^{12} = 1$, and all other components of V^{ab} are 0. Because

$$\begin{aligned} f = \sum_{i<j,k<l} (S_{ik}\delta_{jl} + S_{jl}\delta_{ik} - S_{il}\delta_{jk} - S_{jk}\delta_{il} \\ + 2(\eta t - \epsilon)(\delta_{ik}\delta_{jl} - \epsilon\delta_{il}\delta_{jk}))V^{ij}V^{kl}\,,\end{aligned}$$

at (x_0, t_0), $\nabla_{e_p} f = 0$ is equivalent to

$$\nabla_{e_p} S_{11} + \nabla_{e_p} S_{22} = 2h_{\alpha p1} S_{\alpha 1} + 2h_{\beta p2} S_{\beta 2} = 0\,.$$

Since $S_{n+q,l} = -\frac{2\lambda_q \delta_{ql}}{1+\lambda_q^2}$, we have

$$\frac{\lambda_1}{1+\lambda_1^2} h_{n+1,p1} + \frac{\lambda_2}{1+\lambda_2^2} h_{n+2,p2} = 0 \tag{5.9}$$

for any p.

By (5.3), at (x_0, t_0), we have

$$\left(\frac{d}{dt} - \Delta\right) f = 2\eta + 2R_{k1k\alpha}S_{\alpha 1} + 2R_{k2k\alpha}S_{\alpha 2} + 2h_{\alpha kj}h_{\alpha k1}S_{j1} + 2h_{\alpha kj}h_{\alpha k2}S_{j2} - 2h_{\alpha k1}h_{\beta k1}S_{\alpha\beta} - 2h_{\alpha k2}h_{\beta k2}S_{\alpha\beta}\,. \tag{5.10}$$

The ambient curvature term can be calculated using Lemma 4.2, and we derive

$$\sum_{k,\alpha} R_{k1k\alpha}S_{\alpha 1} + R_{k2k\alpha}S_{\alpha 2} = (k_1 - k_2)(n-1)\sum_{i=1}^{2}\frac{\lambda_i^2}{(1+\lambda_i^2)^2} + (k_1 + k_2)\sum_{i=1}^{2}\frac{\lambda_i^2}{(1+\lambda_i^2)^2}\left[\sum_{j\neq i}\frac{1-\lambda_j^2}{(1+\lambda_j^2)}\right]. \tag{5.11}$$

This can be simplified as

$$\begin{aligned}
&(k_1 - k_2)(n-1)\sum_{i=1}^{2}\frac{\lambda_i^2}{(1+\lambda_i^2)^2}\\
&\quad + (k_1 + k_2)\sum_{i=1}^{2}\frac{\lambda_i^2}{(1+\lambda_i^2)^2}\left[\sum_{j>3}\frac{1-\lambda_j^2}{(1+\lambda_j^2)}\right]\\
&\quad + (k_1 + k_2)\left[\frac{\lambda_1^2}{(1+\lambda_1^2)^2}\frac{1-\lambda_2^2}{(1+\lambda_2^2)} + \frac{\lambda_2^2}{(1+\lambda_2^2)^2}\frac{1-\lambda_1^2}{(1+\lambda_1^2)}\right]\\
&\qquad = (k_1 - k_2)(n-1)\sum_{i=1}^{2}\frac{\lambda_i^2}{(1+\lambda_i^2)^2}\\
&\qquad\quad + (k_1 + k_2)\sum_{i=1}^{2}\frac{\lambda_i^2}{(1+\lambda_i^2)^2}\left[\sum_{j>3}\frac{1-\lambda_j^2}{(1+\lambda_j^2)}\right]\\
&\qquad\quad + (k_1 + k_2)\left[\frac{(\lambda_1^2+\lambda_2^2)(1-\lambda_1^2\lambda_2^2)}{(1+\lambda_1^2)^2(1+\lambda_2^2)^2}\right].
\end{aligned} \tag{5.12}$$

This is nonnegative by equation (5.8).

Using the relations in (4.6) again, the last four terms on the right-hand side of (5.10) can be rewritten as

$$\begin{aligned}
&\sum_{p,k} 2h_{n+p,k1}^2 S_{11} + 2h_{n+p,k2}^2 S_{22} + 2h_{n+p,k1}^2 S_{pp} + 2h_{n+p,k2}^2 S_{pp}\\
&\quad = \sum_k \big(2h_{n+1,k1}^2 S_{11} + 2h_{n+2,k1}^2 S_{11} + 2h_{n+1,k2}^2 S_{22} + 2h_{n+2,k2}^2 S_{22}\\
&\qquad\quad + 2h_{n+1,k1}^2 S_{11} + 2h_{n+2,k1}^2 S_{22} + 2h_{n+1,k2}^2 S_{11} + 2h_{n+2,k2}^2 S_{22}\big)\\
&\qquad + \sum_{q\geq 3,k} 2h_{n+q,k1}^2 S_{11} + 2h_{n+q,k2}^2 S_{22} + 2h_{n+q,k1}^2 S_{qq} + 2h_{n+q,k2}^2 S_{qq}\,.
\end{aligned} \tag{5.13}$$

Since $S_{ii} \geq S_{11}$ for $i \geq 2$, it is clear that (5.13) is nonnegative if $S_{11} \geq 0$. Otherwise, from (5.7), we may assume that

$$S_{11} < 0\,, \quad S_{22} > 0\,, \quad \text{and} \quad S_{11} + S_{22} > 0\,. \tag{5.14}$$

In particular, we have $\lambda_2^2 < \lambda_1^2$ and $\lambda_1^2\lambda_2^2 < 1$. From (5.9), we have

$$h^2_{n+1,p1} = \frac{\lambda_2^2(1+\lambda_1^2)^2}{\lambda_1^2(1+\lambda_2^2)^2} h^2_{n+2,p2}\,.$$

Since $\lambda_2^2 < \lambda_1^2$ and $\lambda_1^2\lambda_2^2 < 1$, we have $\frac{\lambda_2^2(1+\lambda_1^2)^2}{\lambda_1^2(1+\lambda_2^2)^2} < 1$. Thus

$$h^2_{n+1,p1} \leq h^2_{n+2,p2} \quad \text{for all } p \geq 1. \tag{5.15}$$

Recall that $S_{qq} \geq S_{22}$ for $q \geq 3$. The right-hand side of (5.13) can be regrouped as

$$\sum_k \left[(4h^2_{n+1,k1}S_{11} + 4h^2_{n+2,k2}S_{22}) + 2h^2_{n+2,k1}(S_{11}+S_{22}) + 2h^2_{n+1,k2}(S_{11}+S_{22})\right] \\ + \sum_{q\geq 3,k}\left[2h^2_{n+q,k1}(S_{11}+S_{qq}) + 2h^2_{n+q,k2}(S_{22}+S_{qq})\right].$$

This is nonnegative by (5.5), (5.14), and (5.15). Thus, we have $\left(\frac{d}{dt} - \Delta\right) f \geq 2\eta > 0$ at (x_0, t_0), and this is a contradiction. $\square$

Remark. The condition

$$S^{[2]} - \epsilon g^{[2]} \geq 0 \equiv \frac{(1-\lambda_i^2\lambda_j^2)}{(1+\lambda_i^2)(1+\lambda_j^2)} \geq \epsilon \quad \text{for all } i \neq j.$$

In particular, we have $\lambda_i^2 \leq \frac{1-\epsilon}{\epsilon}$. This implies that the Lipschitz norm of f is preserved along the mean curvature flow.

6 Long-Time Existence and Convergence

In this section we prove Theorem 1.1 using the evolution equation (3.4) of $\ln *\Omega$.

PROOF OF THEOREM 1.1: Since $|\lambda_i\lambda_j| < 1$ for $i \neq j$ and Σ_1 is compact, we can find an $\epsilon > 0$ such that

$$\frac{(1-\lambda_i^2\lambda_j^2)}{(1+\lambda_i^2)(1+\lambda_j^2)} \geq \epsilon \quad \text{for all } i \neq j. \tag{6.1}$$

By Lemma 5.3, condition (6.1) is preserved along the mean curvature flow. In particular, we have $|\lambda_i\lambda_j| \leq \sqrt{1-\epsilon}$ and $\lambda_i^2 \leq (1-\epsilon)/\epsilon$. This implies Σ_t remains the graph of a map $f_t : \Sigma_1 \to \Sigma_2$ whenever the flow exists. Each f_t has uniformly bounded $|df_t|$.

We look at the evolution equation (3.4) of $\ln *\Omega$. The quadratic terms of the second fundamental form in equation (3.4) is

$$\sum_{\alpha,i,k} h_{\alpha ik}^2 + \sum_{k,i} \lambda_i^2 h_{n+i,ik}^2 + 2\sum_{k,i<j} \lambda_i\lambda_j h_{n+j,ik} h_{n+i,jk} = \delta|A|^2 + \sum_{k,i} \lambda_i^2 h_{n+i,ik}^2 + (1-\delta)|A|^2 + 2\sum_{k,i<j} \lambda_i\lambda_j h_{n+j,ik} h_{n+i,jk}\,. \tag{6.2}$$

Let $1-\delta = \sqrt{1-\epsilon}$. Using $|\lambda_i\lambda_j| \le 1-\delta$, we conclude that this term is bounded below by $\delta|A|^2$.

By equation (4.11), the curvature term in (3.4) equals

$$\frac{(k_1-k_2)(n-1)}{2}\sum_{i=1}^n \frac{\lambda_i^2}{1+\lambda_i^2} + (k_1+k_2)\sum_{i=1}^n \frac{\lambda_i^2}{1+\lambda_i^2}\Bigg[\sum_{j\neq i}\frac{1-\lambda_j^2}{2(1+\lambda_j^2)}\Bigg]. \tag{6.3}$$

The second term on the right-hand side of (6.3) can be simplified as

$$\begin{aligned}\sum_{i=1}^n \frac{\lambda_i^2}{1+\lambda_i^2}\Bigg[\sum_{j\neq i}\frac{1-\lambda_j^2}{2(1+\lambda_j^2)}\Bigg] &= \sum_{i=1}^n\sum_{i\neq j}\frac{\lambda_i^2-\lambda_i^2\lambda_j^2}{2(1+\lambda_i^2)(1+\lambda_j^2)}\\ &= \sum_{i<j}\frac{\lambda_i^2+\lambda_j^2-2\lambda_i^2\lambda_j^2}{2(1+\lambda_i^2)(1+\lambda_j^2)}\,.\end{aligned} \tag{6.4}$$

This is nonnegative because $|\lambda_i\lambda_j| \le 1-\delta$. Thus $\ln *\Omega$ satisfies the following differential inequality with $k_1 \ge |k_2|$:

$$\frac{d}{dt}\ln *\Omega \ge \Delta \ln *\Omega + \delta|A|^2\,. \tag{6.5}$$

According to the maximum principle for parabolic equations, $\min_{\Sigma_t} \ln *\Omega$ is nondecreasing in time. In particular, $*\Omega \ge \min_{\Sigma_0} *\Omega = \Omega_0$ is preserved and $*\Omega$ has a positive lower bound. Let

$$u = \frac{\ln *\Omega - \ln \Omega_0 + c}{-\ln \Omega_0 + c}$$

where c is a positive number such that $-\ln\Omega_0 + c > 0$. Recall that $0 < *\Omega \le 1$. This implies that $0 < u \le 1$ and u satisfies the differential inequality

$$\frac{d}{dt}u \ge \Delta u + \frac{\delta}{-\ln\Omega_0 + c}|A|^2\,.$$

Because u is also invariant under parabolic dilation, it follows from the blowup analysis in the proof of Theorem 1.1 that the mean curvature flow of the graph of f remains a graph and exists for all time under the assumption that $k_1 \ge |k_2|$ [15].

Using $\lambda_i^2 \le (1-\epsilon)/\epsilon$ and $\lambda_i\lambda_j \le \sqrt{1-\epsilon}$, it is not hard to show

$$(k_1+k_2)\sum_{i<j}\frac{\lambda_i^2+\lambda_j^2-2\lambda_i^2\lambda_j^2}{2(1+\lambda_i^2)(1+\lambda_j^2)} \ge c_1\sum_{i=1}^n \lambda_i^2 \ge c_1 \ln\prod_{i=1}^n \left(1+\lambda_i^2\right), \tag{6.6}$$

where c_1 is a constant that depends on ϵ, k_1, and k_2.

Recall equation (3.3) and we obtain

$$\frac{d}{dt}\ln *\Omega \geq \Delta \ln *\Omega - c_3 \ln *\Omega\,.$$

By the comparison theorem for parabolic equations, $\min_{\Sigma_t} \ln *\Omega$ is nondecreasing in t and $\min_{\Sigma_t} \ln *\Omega \to 0$ as $t \to \infty$. This implies that $\min_{\Sigma_t} *\Omega \to 1$ and $\max |\lambda_i| \to 0$ as $t \to \infty$. We can then apply theorem B in [15] to conclude smooth convergence to a constant map at infinity. □

Acknowledgments. We would like to thank Prof. R. Hamilton, Prof. D. H. Phong, and Prof. S.-T. Yau for their constant advice, encouragement and support. The authors would like to thank the National Center for Theoretical Sciences at National Tsing Hua University in Hsinchu, Taiwan, for the hospitality provided during the preparation of this work. The second author was partially supported by National Science Foundation grants DMS 0104163 and DMS 0306049 and an Alfred P. Sloan Research Fellowship.

Bibliography

[1] Caffarelli, L.; Nirenberg, L.; Spruck, J. The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian. *Acta Math.* **155** (1985), no. 3-4, 261–301.

[2] Chen, H. Pointwise quarter-pinched 4 manifolds. *Ann. Global Anal. Geom.* **9** (1991), 161–176.

[3] Chen, J. Yi; Li, J. Y.; Tian, G. Two-dimensional graphs moving by mean curvature flow. *Acta Math. Sin. (Engl. Ser.)* **18** (2002), no. 2, 209–224.

[4] Ecker, K.; Huisken, G. Mean curvature evolution of entire graphs. *Ann. of Math. (2)* **130** (1989), no. 3, 453–471.

[5] Ecker, K.; Huisken, G. Interior estimates for hypersurfaces moving by mean curvature. *Invent. Math.* **105** (1991), no. 3, 547–569.

[6] Eells, J., Jr.; Sampson, J. H. Harmonic mappings of Riemannian manifolds. *Amer. J. Math.* **86** (1964), 109–160.

[7] Hamilton, R. Four-manifolds with positive curvature operator. *J. Differential Geom.* **24** (1986), no. 2, 153–179.

[8] Hamilton, R. S. Harnack estimate for the mean curvature flow. *J. Differential Geom.* **41** (1995), no. 1, 215–226.

[9] Hamilton, R. S. The formation of singularities in the Ricci flow. *Surveys in differential geometry, vol. II (Cambridge, MA, 1993)*, 7–136. International, Cambridge, Mass., 1995.

[10] Schoen, R. The role of harmonic mappings in rigidity and deformation problems. *Complex geometry (Osaka, 1990)*, 179–200. Lecture Notes in Pure and Applied Mathematics, 143. Dekker, New York, 1993.

[11] Smoczyk, K. Longtime existence of the Lagrangian mean curvature flow. *Calc. Var. Partial Differential Equations* **20** (2004), no. 1, 25–46.

[12] Smoczyk, K.; Wang, M.-T. Mean curvature flows of Lagrangian submanifolds with convex potentials. *J. Differential Geom.* **62** (2002), no. 2, 243–257.

[13] Wang, M.-T. Deforming area preserving diffeomorphism of surfaces by mean curvature flow. *Math. Res. Lett.* **8** (2001), no. 5-6, 651–662.

[14] Wang, M.-T. Mean curvature flow of surfaces in Einstein four-manifolds. *J. Differential Geom.* **57** (2001), no. 2, 301–338.

[15] Wang, M.-T. Long-time existence and convergence of graphic mean curvature flow in arbitrary codimension. *Invent. Math.* **148** (2002), no. 3, 525–543.

[16] Wang, M.-T. Subsets of Grassmannians preserved by mean curvature flow. Preprint, 2002.

[17] Wang, M.-T. On graphic Bernstein type results in higher codimensions. *Trans. Amer. Math. Soc.* **355** (2003), no. 1, 265–271.

[18] Wang, M.-T. Gauss maps of the mean curvature flow. *Math. Res. Lett.* **10** (2003), no. 2-3, 287–299.

[19] Wang, M.-T. Interior gradient bounds for solutions to the minimal surface system. *Amer. J. Math.*, in press.

[20] White, B. A local regularity theorem for classical mean curvature flow. Preprint, 1999 (revised 2002). Available at `http://math.stanford.edu/~white/preprint.htm`.

MAO-PEI TSUI
Columbia University
Department of Mathematics
2990 Broadway
Mail Code: 4406
New York, NY 10027
E-mail: tsui@math.columbia.edu

MU-TAO WANG
Columbia University
Department of Mathematics
2990 Broadway
Mail Code: 4406
New York, NY 10027
E-mail: mtwang@math.columbia.edu

Received August 2003.