Review Problems for Final Exam

1. Let S be the triangle with vertices $A = (2, 2, 2)$, $B = (4, 2, 1)$ and $C = (2, 3, 1)$.

(a) Find the cosine of the angle BAC at vertex A.

(b) Find the area of the triangle ABC.

(c) Find a vector that is perpendicular to the plane that contains the points A, B and C.

(d) Find the equation of the plane through A, B and C.

(e) Find the distance between $D = (3, 1, 1)$ and the plane through A, B and C.

(f) Find the volume of the parallelepiped formed by $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$.

2. Find the distance between the planes $2x - y + 2z = 10$ and $4x - 2y + 4z = 7$.

3. **(a)** Find a vector equation of the line through $(2, 4, 1)$ and $(4, 5, 3)$

(b) Find a vector equation of the line through $(1, 1, 1)$ that is parallel to the line through $(2, 4, 1)$ and $(4, 5, 3)$.

(c) Find a vector equation of the line through $(1, 1, 1)$ that is parallel to the line $\frac{x-2}{2} = -\frac{y}{1} = \frac{z-2}{2}$.

4. **(a)** Find the equation of a plane perpendicular to the vector $\vec{i} - \vec{j} + \vec{k}$ and passing through the point $(1, 1, 1)$.

(b) Find the equation of a plane perpendicular to the planes $3x + 2y - z = 7$ and $x - 4y + 2z = 0$ and passing through the point $(1, 1, 1)$.

5. Find the arc-length of the curve $r(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$ when $0 \leq t \leq \ln(2)$.

6. Find parametric equations for the tangent line to the curve $r(t) = \langle t^3, t, t^3 \rangle$ at the point $(-1, 1, -1)$.

7. Find the linear approximation of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at $(1, 2, 2)$ and use it to estimate $\sqrt{(1.1)^2 + (2.1)^2 + (1.9)^2}$.

8. **(a)** Find the equation for the plane tangent to the surface $z = 3x^2 - y^2 + 2x$ at $(1, -2, 1)$.

(b) Find the equation for the plane tangent to the surface $x^2 + xy^2 + xyz = 4$ at $(1, 1, 2)$.

9. Suppose that over a certain region of plane the electrical potential is given by $V(x, y) = x^2 - xy + y^2$.

(a) Find the direction of the greatest decrease in the electrical potential at the point $(1, 1)$. What is the magnitude of the greatest decrease?

(b) Find the rate of change of V at $(1, 1)$ in the direction $\langle 3, -4 \rangle$.

10. Find the local maxima, local minima and saddle points of the following functions. Decide if the local maxima or minima is global maxima or minima. Explain.

(a) $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2$

(b) $f(x, y) = x^2 + y^3 - 3xy$

11. Use Lagrange multipliers to find the maximum or minimum values of f subject to the given constraint.

(a) $f(x, y, z) = x^2 - y^2$, $x^2 + y^2 = 2$

(b) $f(x, y, z) = x + y + z$, $x^2 + y^2 + z^2 = 1$.

12. Compute the following iterated integrals.

(a) $\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3} dxdy$

(b) $\int_0^2 \int_{-\sqrt{4-x^2}}^0 e^{-x^2-y^2} dydx$

(c) $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} (x^2+y^2)^{\frac{3}{2}} dzdydx$

(d) $\int_{-2}^2 \int_0^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2+y^2+z^2} dzdxdy$

13. Find the volume of the following regions:

(a) The solid bounded by the surface $z = x\sqrt{x^2+y}$ and the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$ and $z = 0$.

(b) The solid bounded by the plane $x + y + z = 3$, $x = 0$, $y = 0$ and $z = 0$.

(c) The region bounded by the cylinder $x^2 + y^2 = 4$ and the plane $z = 0$ and $y + z = 3$.

14. Let C be the oriented path which is a straight line segment running from $(1, 1, 1)$ to $(0, -1, 3)$. Calculate $\int f ds$ where $f = (x + y + z)$.

15. Calculate the following line integrals $\int_C \vec{F} \cdot d\vec{r}$:

(a) $\vec{F} = y\sin(xy)\vec{i} + x\sin(xy)\vec{j}$ and C is the parabola $y = 2x^2$ from $(1, 2)$ to $(3, 18)$.

(b) $\vec{F} = 2x\vec{i} - 4y\vec{j} + (2z-3)\vec{k}$ and C is the line from $(1, 1, 1)$ to $(2, 3, -1)$.

16. Calculate the circulation of $\vec{F}$ around the given paths.

(a) $\vec{F} = xy\vec{j}$ around the square $0 \le x \le 1$, $0 \le y \le 1$ oriented counterclockwise.

(b) $\vec{F} = (2x^2+3y)\vec{i} + (2x+3y^2)\vec{j}$ around the triangle with vertices $(2, 0)$, $(0, 3)$, $(-2, 0)$ oriented counterclockwise.

(c) $\vec{F} = 3y\vec{i} + xy\vec{j}$ around the unit circle oriented counterclockwise.

(d) $\vec{F} = xz\vec{i} + (x+yz)\vec{j} + x^2\vec{k}$ and C is the circle $x^2 + y^2 = 1$, $z = 2$ oriented counterclockwise when viewed from above.

17. Calculate the area of the region within the ellipse $x^2/a^2 + y^2/b^2 = 1$ parameterized by $x = a\cos(t)$, $y = b\sin(t)$ for $0 \le t \le 2\pi$.

18. Compute the flux of the vector field $\vec{F}$ through the surface S.

(a) $\vec{F} = x\vec{i} + y\vec{j}$ and S is the part of the surface $z = 25 - (x^2+y^2)$ above the disk of radius 5 centered at the origin oriented upward.

(b) $\vec{F} = -y\vec{i} + z\vec{k}$ and S is the part of the surface $z = y^2 + 5$ over the rectangle $-2 \le x \le 1$, $0 \le y \le 1$ oriented upward.

(c) $\vec{F} = y\vec{i} + \vec{j} - xz\vec{k}$ and S is the surface $y = x^2 + z^2$ with $x^2 + z^2 \leq 1$ oriented in the positive y-direction.

(d) $\vec{F} = x^2\vec{i} + (y - 2xy)\vec{j} + 10z\vec{k}$ and S is the sphere of radius 5 centered at the origin oriented outward.

(e) $\vec{F} = -z\vec{i} + x\vec{k}$ and S is a square pyramid with height 3 and base on the xy-plane of side length 1.

(f) $\vec{F} = y\vec{j}$ and S is a closed vertical cylinder of height 2 with its base a circle of radius 1 on the xy-plane centered at the origin.

19. Let $\vec{F} = (8yz - z)\vec{j} + (3 - 4z^2)\vec{k}$.

(a) Show that $\vec{G} = 4yz^2\vec{i} + 3x\vec{j} + xz\vec{k}$ is a vector potential for $\vec{F}$.

(b) Evaluate $\int_S \vec{F} \cdot d\vec{A}$ where S is the upper hemisphere of radius 5 centered at the origin oriented upwards.

20. For constants a, b, c and m consider the vector field

$$\vec{F} = (ax + by + 5z)\vec{i} + (x + cz)\vec{j} + (3y + mx)\vec{k}\,.$$

(a) Suppose that the flux of $\vec{F}$ through any closed surface is 0. What does this tell you about the values of the constants a, b, c, m?

(b) Suppose instead that the circulation of $\vec{F}$ around any closed curve is 0. What does this tell you about the values of the constants a, b, c, m?