Solutions for Review Problems

1. Let S be the triangle with vertices $A = (2, 2, 2)$, $B = (4, 2, 1)$ and $C = (2, 3, 1)$.

(a) Find the cosine of the angle BAC at vertex A.

(b) Find the area of the triangle ABC.

(c) Find a vector that is perpendicular to the plane that contains the points A, B and C.

(d) Find the equation of the plane through A, B and C.

(e) Find the distance between $D = (3, 1, 1)$ and the plane through A, B and C.

(f) Find the volume of the parallelepiped formed by $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$.

Solution. **(a)** Using $A = (2, 2, 2)$, $B = (4, 2, 1)$ and $C = (2, 3, 1)$, We have $\vec{AB} =< 2, 0, -1 >$, $\vec{AC} =< 0, 1, -1 >$ and $\vec{BC} =< -2, 1, 0 >$.

Let θ be the angle BAC at vertex A.
We have $\cos(\theta) = \frac{\vec{AB}\cdot\vec{AC}}{||\vec{AB}||\cdot||\vec{AC}||} = \frac{<2,0,-1>\cdot<0,1,-1>}{\sqrt{5}\cdot\sqrt{2}} = \frac{1}{\sqrt{10}}$.

(b) First we find

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \vec{k} = \vec{i} + 2\vec{j} + 2\vec{k} =< 1, 2, 2 >$$

Thus the area of the triangle $ABC = \frac{1}{2}||\vec{AB} \times \vec{AC}|| = \frac{1}{2}\sqrt{2^2 + 2^2 + 1} = \frac{3}{2}$.

(c) The vector $\vec{AB} \times \vec{AC} =< 1, 2, 2 >$ is perpendicular to the plane that contains the points A, B and C.

(d) The normal vector of the plane is $\vec{AB} \times \vec{AC} =< 1, 2, 2 >$. So the equation of the plane thru $A = (2, 2, 2)$ with normal vector $< 1, 2, 2 >$ is $< x-2, y-2, z-2 > \cdot < 1, 2, 2 >= 0$, i.e. $x - 2 + 2y - 4 + 2z - 4 = 0$. So the plane thru A, B and C is $x + 2y + 2z = 10$.

(e) From previous problem, we know that the equation of the plane through A, B and C is $x + 2y + 2z = 10$. So the distance between $D = (3, 1, 1)$ and the plane $x + 2y + 2z = 10$ is $\frac{|3+2\cdot1+2\cdot1-10|}{\sqrt{1^2+2^2+2^2}} = \frac{3}{3} = 1$.

(f) The volume of the parallelepiped formed by $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$
$= |(\vec{AB} \times \vec{AC}) \cdot \vec{AD}| = | - 2| = 2$.

□

2. Find the distance between the planes $2x - y + 2z = 10$ and $4x - 2y + 4z = 7$.

Solution. The plane $4x - 2y + 4z = 7$ can be rewritten as $2x - y + 2z = \frac{7}{2}$. Using the distance formula between planes, the distance between $P_1 : 2x - y + 2z = 10$

and $P_2 : 2x - y + 2z = \frac{7}{2}$ is $\frac{|10-\frac{7}{2}|}{\sqrt{2^2+2^2+(-1)^2}} = \frac{13}{6}$.

A plane P is drawn through the points $A = (1,-1,0)$, $B = (0,1,-1)$ and $C = (1,0,-1)$. Find the distance between the plane P and the point $(1,1,1)$.

$\square$

3. **(a)** Find a vector equation of the line through $(2,4,1)$ and $(4,5,3)$

(b) Find a vector equation of the line through $(1,1,1)$ that is parallel to the line through $(2,4,1)$ and $(4,5,3)$.

(c) Find a vector equation of the line through $(1,1,1)$ that is parallel to the line $\frac{x-2}{2} = -\frac{y}{1} = \frac{z-2}{2}$.

Solution.

(a) Let $P = (2,4,1)$ and $Q = (4,5,3)$. The direction of the line is $\overrightarrow{PQ} =< 4-2, 5-4, 3-1 >=< 2,1,2 >$. The vector equation is $< x,y,z >=< 2,4,1 > +t < 2,1,2 >$ or $x = 2+2t$, $y = 4+t$ and $z = 1+2t$.

(b) The direction of the line is $\overrightarrow{PQ} =< 4-2, 5-4, 3-1 >=< 2,1,2 >$. The starting point of the line is $(1,1,1)$. So the equation of the line is $x = 1+2t$, $y = 1+t$ and $z = 1+2t$.

(c) The direction of the line $\frac{x-2}{2} = -\frac{y}{1} = \frac{z-2}{2}$ is $< 2,-1,2 >$. The starting point of the line is $(1,1,1)$. So the equation of the line is $x = 1+2t$, $y = 1-t$ and $z = 1+2t$.

$\square$

4. **(a)** Find the equation of a plane perpendicular to the vector $\vec{i} - \vec{j} + \vec{k}$ and passing through the point $(1,1,1)$.

(b) Find the equation of a plane perpendicular to the planes $3x+2y-z=7$ and $x-4y+2z=0$ and passing through the point $(1,1,1)$.

Solution.

(a) The equation of the plane with normal vector $\vec{i}-\vec{j}+\vec{k}$ and passing through the point $(1,1,1)$ is
$\langle 1,-1,1\rangle \cdot \langle x-1, y-1, z-1\rangle = (x-1)-(y-1)+(z-1) = 0$ or $x-y+z=1$.

(b) The plane that is perpendicular to the planes $3x+2y-z=7$ and $x-4y+2z=0$ has normal vector
$\langle 3,2,-1\rangle \times \langle 1,-4,2\rangle = \langle 0,-7,-14\rangle$. Thus the equation of the plane is $-7y-14z=-21$, that is $y+2z=3$.

$\square$

5. Find the arc-length of the curve $r(t) = \langle \sqrt{2}t, e^t, e^{-t}\rangle$ when $0 \le t \le \ln(2)$.

Solution. Given $r(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$, we have $r'(t) = \langle \sqrt{2}, e^t, -e^{-t} \rangle$ and $|r'(t)| = \sqrt{2 + e^{-2t} + e^{2t}} == \sqrt{(e^{-t} + e^t)^2} = e^{-t} + e^t$. Hence the arc-length of the curve $r(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$ between $0 \le t \le \ln(2)$ is $\int_0^{\ln(2)} |r'(t)| dt = \int_0^{\ln(2)} (e^{-t} + e^t) dt = -e^{-t} + e^t|_0^{\ln(2)} = -e^{-\ln(2)} + e^{\ln(2)} - (-1 + 1) = -\frac{1}{2} + 2 = \frac{3}{2}$. Note that $e^{-\ln(2)} = \frac{1}{e^{\ln(2)}} = \frac{1}{2}$.

□

6. Find parametric equations for the tangent line to the curve $r(t) = \langle t^3, t, t^3 \rangle$ at the point $(-1, 1, -1)$.

Solution. Note that $r(t) = \langle t^3, t, t^3 \rangle$. We have $r(-1) = \langle -1, 1, -1 \rangle$. Taking the derivative of $r(t)$, we get $r'(t) = \langle 3t^2, 1, 3t^3 \rangle$. Thus the tangent vector at $t = -1$ is $r'(-1) = \langle 3, 1, 3 \rangle$. Therefore parametric equations for the tangent line is $x = -1 + 3t$, $y = 1 + t$ and $z = -1 + 3t$.

□

7. Find the linear approximation of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at $(1, 2, 2)$ and use it to estimate $\sqrt{(1.1)^2 + (2.1)^2 + (1.9)^2}$.

Solution. The partial derivatives are $f_x(x, y, z) = \frac{x}{\sqrt{x^2+y^2+z^2}}$, $f_y(x, y, z) = \frac{y}{\sqrt{x^2+y^2+z^2}}$, $f_y(x, y, z) = \frac{z}{\sqrt{x^2+y^2+z^2}}$, $f_x(1, 2, 2) = \frac{1}{3}$ and $f_y(1, 2, 2) = \frac{2}{3}$ and $f_z(1, 2, 2) = \frac{2}{3}$.
The linear approximation of $f(x, y, z)$ at $(1, 2, 2)$ is

$$\begin{aligned} L(x, y, z) &= f(1, 2, 2) + f_x(1, 2, 2)(x - 1) + f_y(1, 2, 2)(y - 2) + f_z(1, 2, 2)(z - 2) \\ &= 3 + \frac{1}{3}(x - 1) + \frac{2}{3}(y - 2) + \frac{2}{3}(z - 2). \end{aligned}$$

Thus $L(1.1, 2.1, 1.9) = 3 + \frac{1}{3}(1.1 - 1) + \frac{2}{3}(2.1 - 2) + \frac{2}{3}(1.9 - 2) = 3 + \frac{.1+.2-.2}{3} = 3 + \frac{.1}{3} \approx 3.033$. Hence $\sqrt{(1.1)^2 + (2.1)^2 + (1.9)^2}$ is about 3.033.

□

8. **(a)** Find the equation for the plane tangent to the surface $z = 3x^2 - y^2 + 2x$ at $(1, -2, 1)$.

(b) Find the equation for the plane tangent to the surface $x^2 + xy^2 + xyz = 4$ at $(1, 1, 2)$.

Solution. (a) Let $f(x, y) = 3x^2 - y^2 + 2x$. We have $f_x = 6x + 2$, $f_y = -2y$, $f_x(1, -2) = 8$ and $f_y(1, -2) = 4$. The equation of the tangent plane through the point $(1, -2, 1)$ is

$$\begin{aligned} z &= f(1, -2) + f_x(1, -2)(x - 1) + f_y(1, -2)(y + 2) \\ &= 1 + 8(x - 1) + 4(y + 2) = 8x + 4y + 1. \end{aligned}$$

(b) In general, the normal vector for the tangent plane to the level surface of $F(x, y, z) = k$ at the point (a, b, c) is $\nabla F(a, b, c)$.

The surface $x^2+xy^2+xyz = 4$ can be rewritten as $F(x, y, z) = x^2+xy^2+xyz = 4$, $\nabla F(x, y, z) = \langle 2x + y^2 + yz, 2xy + xz, xy\rangle$ and
$\nabla F(1, 1, 2) = \langle 5, 4, 1\rangle$ Thus the equation of the tangent plane to the surface $x^2 + xy^2 + xyz = 4$ at the point $(1, 1, 2)$ is
$\langle 5, 4, 1\rangle \cdot \langle x - 1, y - 1, z - 2\rangle = 0$ which yields
$5x - 5 + 4y - 4 + z - 2 = 0$. It can be simplified as $5x + 4y + z - 11 = 0$.

□

9. Suppose that over a certain region of plane the electrical potential is given by $V(x, y) = x^2 - xy + y^2$.

(a) Find the direction of the greatest decrease in the electrical potential at the point $(1, 1)$. What is the magnitude of the greatest decrease?

(b) Find the rate of change of V at $(1, 1)$ in the direction $\langle 3, -4\rangle$.

Solution. (a) We have

$$\nabla V(x, y) = \langle V_x(x, y), V_y(x, y)\rangle = \langle (x^2 - xy + y^2)_x, (x^2 - xy + y^2)_y\rangle = \langle 2x - y, -x + 2y\rangle$$

Since

$$\nabla V(x, y) = \langle 2x - y, -x + 2y\rangle$$

the direction of the greatest decrease in electrical potential is

$$-\nabla V(1, 1) = -\langle 1, 1\rangle$$

and the magnitude is $-\|\nabla V(1, 1)\| = -\sqrt{2}$.

(b) The unit vector in the direction $\langle 3, -4\rangle$ is $\vec{u} = \frac{1}{5}\langle 3, -4\rangle$. Thus the rate of change of V at $(1, 1)$ in the direction $\langle 3, -4\rangle$ is

$$\nabla V(1, 1) \cdot \vec{u} = \langle 1, 1\rangle \cdot \frac{1}{5}\langle 3, -4\rangle = -\frac{1}{5}.$$

□

10. Find the local maxima, local minima and saddle points of the following functions. Decide if the local maxima or minima is global maxima or minima. Explain.

(a) $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2$

(b) $f(x, y) = x^2 + y^3 - 3xy$

Solution. (a) To find critical points, set $f_x(x,y) = 12 - 6x = 0$ and $f_y(x,y) = 6 - 2y = 0$. Hence, $(2,3)$ is the only critical point. We also have $f_{xx} = -6$, $f_{xy} = f_{yx} = 0$ and $f_{yy} = -2$.

$$(D^2 f(x,y)) = (\begin{array}{cc} -6 & 0 \\ 0 & -2 \end{array})$$

Since $det(D^2 f(2,3)) = 12 > 0$ and $f_{xx}(2,3) < 0$, the second derivative test implies that f has a local maximum at $(2,3)$. Because f is a quadratic function, it follows the graph of f is an elliptical paraboloid and $(2,3)$ is a global maximum. We can also see that f has a global maximum at $(2,3)$ be completing the square: $f(x,y) = 31 - 3(x-2)^2 - (y-3)^2$.

(b) The system of equations

$$f_x(x,y) = 2x - 3y = 0 \qquad f_y(x,y) = 3y^2 - 3x = 0$$

implies that $x = \frac{3}{2}y$ and $3\big(y^2 - \frac{3}{2}y\big) = 2y\big(y - \frac{3}{2}\big) = 0$. Thus, $(0,0)$ and $(9/4, 3/2)$ are the critical points. We also have $f_{xx} = 2$, $f_{xy} = f_{yx} = -3$ and $f_{yy} = 6y$.

$$(D^2 f(x,y)) = \begin{pmatrix} 2 & -3 \\ -3 & 6y \end{pmatrix}$$

Since

$$\begin{aligned} Det(D^2 f(x,y)) &= f_{xx} f_{yy} - f_{xy}^2 = (2)(6y) - (-3)^2 = 12y - 9, \\ Det(D^2 f(0,0)) &= -9 < 0, \\ Det(D^2 f(9/4, 3/2)) &= 18 > 0, \end{aligned}$$

the second derivative test establishes that f has a saddle point at $(0,0)$ and a local minimum at $(9/4, 3/2)$. Because $\lim_{y\to-\infty} f(0,y) = \lim_{y\to-\infty} y^3 = -\infty$, we see that $(9/4, 3/2)$ is not a global minimum.

□

11. Use Lagrange multipliers to find the maximum or minimum values of f subject to the given constraint.

(a) $f(x,y,z) = x^2 - y^2$, $x^2 + y^2 = 2$

Solution. Let $f(x,y) = x^2 - y^2$ and $g_(x,y) = x^2 + y^2 = 2$. The necessary conditions for the optimizer (x,y) are
$\nabla f(x,y) = \lambda \nabla g_(x,y)$ and the constraint equations $x^2 + y^2 = 2$ which are: Since $\nabla f(x,y) = (2x, -2y)$ and $\nabla g(x,y) = (2x, 2y)$, thus (x,y) must satisfy

$$2x = 2\lambda x \tag{0.0.1}$$
$$-2y = 2\lambda y \tag{0.0.2}$$
$$x^2 + y^2 = 2 \tag{0.0.3}$$

From (4), (5) , we get $4x^2 + 4y^2 = 4\lambda^2(x^2 + y^2)$. Since $x^2 + y^2 = 2$m we have $\lambda^2 = 1$. So $\lambda = \pm 1$. If $lambda = 1$, then eq(4) is always true and we get $y = 0$ by eq(5). Using $x^2 + y^2 = 2$, we get $x = \pm\sqrt{2}$. If $lambda = -1$, then eq(5) is always true and we get $x = 0$ by eq(4). Using $x^2 + y^2 = 2$, we get $y = \pm\sqrt{2}$.
So the candidates are $(\sqrt{2}, 0)$, $(-\sqrt{2}, 0)$, $(0, \sqrt{2}, 0)$ and $(0, \sqrt{2}, 0)$.
So $f((\sqrt{2}, 0)) = f((-\sqrt{2}, 0)) = 2$ and $f((0, \sqrt{2}, 0)) = f((0, \sqrt{2}, 0)) = -2$.
Thus the maximum is 2, the minimum is -2, the maximizers are $(\sqrt{2}, 0)$, $(-\sqrt{2}, 0)$, and the minimizers are $(0, \sqrt{2}, 0)$ and $(0, \sqrt{2}, 0)$.

□

(b) $f(x, y, z) = x + y + z$, $x^2 + y^2 + z^2 = 1$.

Solution. Let $f(x, y, z) = x + y + z$ and $g(x, y, z) = x^2 + y^2 + z^2 = 1$.
We have $\nabla f(x, y, z) = (1, 1, 1)$ and$\nabla g(x, y, z) = (2x, 2y, 2z)$.
The necessary conditions for the optimizer (x, y, z) are $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and the constraint equations which are:

$$1 = 2\lambda x \tag{0.0.4}$$
$$1 = 2\lambda y \tag{0.0.5}$$
$$1 = 2\lambda z \tag{0.0.6}$$
$$x^2 + y^2 + z^2 = 1 \tag{0.0.7}$$

From (7),(8) (9) and (10), we know that $\lambda \neq 0$, $x = \frac{1}{2\lambda}$, $y = \frac{1}{2\lambda}$ and $z = \frac{1}{2\lambda}$. Plugging into (10), we get $\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 1$, $\frac{3}{4\lambda^2} = 1$ and $\lambda = \pm\frac{\sqrt{3}}{2}$. So $(x, y, z) = (\frac{1}{2\lambda}, \frac{1}{2\lambda}, \frac{1}{2\lambda}) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ or $(x, y, z) = ((-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.
We have $f((\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})) = \frac{3}{\sqrt{3}} = \sqrt{3}$ and $f((-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})) = -\frac{3}{\sqrt{3}} = -\sqrt{3}$.
Thus the maximizers are $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ with maximum $\sqrt{3}$. The minimizers are $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ with minimum $-\sqrt{3}$. □

12. Compute the following iterated integrals.

(a) $\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3} dxdy$

Let $D = \{(x, y) | \sqrt{y} \leq x \leq 1, 0 \leq y \leq 1\}$ Then $0 \leq y \leq x^2$ and $0 \leq x \leq 1$. So D is the same as $\{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq x^2\}$.
We have $\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3} dxdy = \int_0^1 \int_0^{x^2} \frac{ye^{x^2}}{x^3} dydx = \int_0^1 \frac{y^2 e^{x^2}}{2x^3}\Big|_0^{x^2} dx = \int_0^1 \frac{xe^{x^2}}{2} dx = \frac{e^{x^2}}{4}|_0^1 = \frac{e}{4} - \frac{1}{4}$.

(b) $\int_0^2 \int_{-\sqrt{4-x^2}}^0 e^{-x^2-y^2} dydx$

Solution. The region of integration is $\{(x,y)0 \le x \le 2, -\sqrt{4-x^2} \le y \le 0\}$. The is the region in fourth quadrant. In polar coordinates, it is $R = \left\{(r,\theta) : 0 \le r \le 3, \frac{-\pi}{2} \le \theta \le 0\right\}$. We also have $x^2+y^2 = r^2$ and
$\int_0^2 \int_{-\sqrt{4-x^2}}^0 e^{-x^2-y^2} dydx = \int_{\frac{-\pi}{2}}^0 \int_0^2 e^{-r^2} \cdot r dr d\theta$
$= \int_{\frac{-\pi}{2}}^0 -\frac{e^{-r^2}}{2}\Big|_0^2 d\theta = -(\frac{e^{-4}}{2} - \frac{1}{2}) \cdot \frac{\pi}{2}$. □

(c) $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} (x^2+y^2)^{\frac{3}{2}} dzdydx$

Solution. The region of integration is $\{(x,y,z)| -1 \le x \le 1, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, x^2+y^2 \le z \le 2-x^2-y^2\}$. In cylindrical coordinates, it is $R = \left\{(r,\theta,z) : 0 \le r \le 1, 0 \le \theta \le 2\pi, r^2 \le z \le 2-r^2\right\}$. Recall that $x = r\cos(\theta)$, $x = r\sin(\theta)$ We have $(x^2+y^2)^{\frac{3}{2}} = r^3$ and
$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} (x^2+y^2)^{\frac{3}{2}} dzdydx = \int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} r^3 \cdot r dz dr d\theta$
$= \int_0^{2\pi} \int_0^1 r^4 z\Big|_{r^2}^{2-r^2} dr d\theta$
$= \int_0^{2\pi} \int_0^1 r^4(2-2r^2) dr d\theta$
$= \int_0^{2\pi} \int_0^1 (\frac{2r^5}{5} - \frac{2r^7}{7})\Big|_0^1 d\theta$
$= \int_0^{\frac{\pi}{2}} \frac{4}{35} d\theta$
$= \frac{8\pi}{35}$.

□

(d) $\int_{-2}^2 \int_0^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2+y^2+z^2} dzdxdy$

Solution. In spherical coordinates, the region $E = \{(x,y,z)|0 \le x \le \sqrt{4-y^2}, -2 \le y \le 2, -\sqrt{4-x^2-y^2} \le z \le \sqrt{4-x^2-y^2}\}$

is described by the inequalities $0 \le \rho \le 2$, $0 \le \theta \le \pi\pi$ and $0 \le \phi \le \pi$. Note that $y = \rho\sin(\phi)\cos(\theta)$ Hence, the integral is

$$\begin{aligned}
&\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}dzdxdy \\
&= \int_0^{\pi}\int_0^{\pi}\int_0^{2} \rho^2\sin^2(\phi)\cos^2(\theta)(\rho)\ \rho^2\sin(\phi)\ d\rho\ d\theta\ d\phi \\
&= \int_0^{\pi}\int_0^{\pi}\int_0^{2} \rho^5\sin^3(\phi)\cos^2(\theta)\ d\rho\ d\theta\ d\phi \\
&= \left(\int_0^{\pi}\cos^2(\theta)d\theta\right)\left(\int_0^{\pi}\sin^3(\phi)\ d\phi\right)\left(\int_0^{2}\rho^5\ d\rho\right) \\
&= \left(\int_0^{\pi}\frac{1+\cos(2\theta)}{2}d\theta\right)\left(\int_0^{\pi}(1-\cos^2(\phi))\sin(\phi)\ d\phi\right)\left(\int_0^{2}\rho^5\ d\rho\right) \\
&= \left((\frac{\theta}{2}+\frac{\sin(2\theta)}{4})\Big|_0^{\pi}\right)\left((-\cos(\phi)+\frac{\cos^3(\phi)}{3})\Big|_0^{\pi}\right)\left(\frac{\rho^6}{6}\Big|_0^{2}\right) \\
&= \frac{\pi}{2}\cdot\frac{4}{3}\cdot\frac{64}{6} = \frac{64\pi}{9}
\end{aligned}$$

□

13. Find the volume of the following regions:

(a) The solid bounded by the surface $z = x\sqrt{x^2+y}$ and the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$ and $z = 0$.

Solution. The volume is $\int_0^1\int_0^1 x\sqrt{x^2+y}dxdy$ Let $u = x^2+y$. Then $du = 2xdx$, $xdx = \frac{du}{2}$ and $\int x\sqrt{x^2+y}dx = \int \frac{u^{1/2}}{2}du = \frac{u^{3/2}}{3} + C = \frac{(x^2+y)^{3/2}}{3} + C$. So $\int_0^1\int_0^1 x\sqrt{x^2+y}dxdy = \int_0^1 \frac{(x^2+y)^{3/2}}{3}\Big|_0^1 dy$
$= \int_0^1 \frac{(1+y)^{3/2}}{3} - \frac{(y)^{3/2}}{3}dx = \frac{2(1+y)^{5/2}}{15} - \frac{2(y)^{5/2}}{15}\Big|_0^1 = \frac{2(2)^{5/2}}{15} - \frac{2}{15} - (\frac{2}{15} - 0)$
$= \frac{2(2)^{5/2}}{15} - \frac{4}{15} = \frac{8\sqrt{2}}{15} - \frac{4}{15}$ □

(b) The solid bounded by the plane $x+y+z = 3$, $x = 0$, $y = 0$ and $z = 0$.

Solution. The region E bounded by the xy, yz, xz planes and the plane $x+y+z = 3$ is the set $\{(x, y, z) \in \mathbb{R}^3 : 0 \le x \le 3, 0 \le y \le 3-x, 0 \le z \le$

$3-x-y\}$. The volume of E is

$$\iiint_E dV = \int_0^{3)}\int_0^{3-x}\int_0^{3-x-y} dz\,dy\,dx = \int_0^{3)}\int_0^{3-x} z\Big|_0^{3-x-y} dy\,dx$$

$$= \int_0^3\int_0^{3-x} 3-x-y\,dy\,dx = \int_0^3 3y-xy-\frac{y^2}{2}\Big|_0^{3-x} dx \text{ (by substitution u=4-x-2y)}$$

$$= \int_0^3 3(3-x)-x(3-x)-\frac{(3-x)^2}{2}\,dx = \int_0^3 9-3x+3-3x+x^2-\frac{(x^2-6x+9)}{2}\,dx$$

$$= \int_0^3 \frac{9}{2}-3x+\frac{x^2}{2}\,dx = \frac{9x}{2}-\frac{3x^2}{2}+\frac{x^3}{6}\Big|_0^3 = \frac{27}{2}-\frac{27}{2}+\frac{27}{6}=\frac{9}{2}.$$

□

(c) The region bounded by the cylinder $x^2+y^2=4$ and the plane $z=0$ and $y+z=3$.

Solution. The region is bounded above by the plane $z=3-y$ and below by $z=0$. In polar coordinates, this region $x^2+y^2\le 4$ is $R=\{(r,\theta): 0\le r\le 2, 0\le\theta\le 2\pi\}$. Note that $z=3-y=3-r\cos(\theta)$ Hence, we can compute the volume of the region by

$$\begin{aligned}\text{Volume} &= \int_0^{2\pi}\int_0^2 \big(3-r\cos(\theta)\big)\,r dr\,d\theta\\ &= \int_0^{2\pi}\int_0^2 3r-r^2\cos(\theta)\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{3}{2}r^2-\tfrac{1}{3}r^3\cos(\theta)\right]_0^2 d\theta\\ &= \int_0^{2\pi}\left[6-\tfrac{8}{3}\cos(\theta)\right]d\theta = 12\pi\,.\end{aligned}$$

□

14. Let C be the oriented path which is a straight line segment running from $(1,1,1)$ to $(0,-1,3)$. Calculate $\int f ds$ where $f=(x+y+z)$.

Solution.
C is parametrized by $x(t)=1-t$, $y(t)=1-2t$ and $z(t)=1+2t$ where $0\le t\le 1$. We have $f(x(t),y(t),z(t))=x(t)+y(t)+z(t)=1-t+1-2t+1+2t=3-t$ and $ds=\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}dt=\sqrt{(-1)^2+(-2)^2+2^2}dt=\sqrt{9}dt=3dt$. So $\int_C f ds=\int_0^1(3-t)dt=3t-\frac{t^2}{2}|_0^1=\frac{5}{2}$.

□

15. Calculate the following line integrals $\int_C \vec{F}\cdot d\vec{r}$:

(a) $\vec{F}=y\sin(xy)\vec{i}+x\sin(xy)\vec{j}$ and C is the parabola $y=2x^2$ from $(1,2)$ to $(3,18)$.

(b) $\vec{F}=2x\vec{i}-4y\vec{j}+(2z-3)\vec{k}$ and C is the line from $(1,1,1)$ to $(2,3,-1)$.

Solution.

(a) If $f(x,y) = -\cos(xy)$ then $\nabla f = y\sin(xy)\vec{i} + x\sin(xy)\vec{j} = \vec{F}$. Hence, the Fundamental Theorem for line integrals implies that

$$\int_C \vec{F}\cdot d\vec{r} = f(3,18) - f(1,2) = \cos(2) - \cos(54)\,.$$

(b) If $f(x,y,z) = x^2 - 2y^2 + z^2 - 3z$ then $\nabla f = 2x\vec{i} - 4y\vec{j} + (2z-3)\vec{k} = \vec{F}$. Hence, the Fundamental Theorem for line integrals implies that

$$\int_C \vec{F}\cdot d\vec{r} = f(2,3,-1) - f(1,1,1) = -10 + 3 = -7\,.$$

□

16. Calculate the circulation of $\vec{F}$ around the given paths.

(a) $\vec{F} = xy\vec{j}$ around the square $0 \le x \le 1$, $0 \le y \le 1$ oriented counterclockwise.

(b) $\vec{F} = (2x^2+3y)\vec{i} + (2x+3y^2)\vec{j}$ around the triangle with vertices $(2,0)$, $(0,3)$, $(-2,0)$ oriented counterclockwise.

(c) $\vec{F} = 3y\vec{i} + xy\vec{j}$ around the unit circle oriented counterclockwise.

(d) $\vec{F} = xz\vec{i} + (x+yz)\vec{j} + x^2\vec{k}$ and C is the circle $x^2+y^2 = 1$, $z = 2$ oriented counterclockwise when viewed from above.

Solution.

(a) If $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$ then Green's Theorem implies that

$$\int_{\partial R} \vec{F}\cdot d\vec{r} = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \int_0^1\int_0^1 y\;dy\;dx = \left(\int_0^1 dx\right)\left(\int_0^1 y\;dy\right) = \frac{1}{2}\,.$$

(b) If T is the triangle with vertices $(2,0)$, $(0,3)$, $(-2,0)$ then Green's Theorem gives

$$\int_{\partial T} \vec{F}\cdot d\vec{r} = \int_T \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \int_T 2 - 3\;dA = -\text{Area}(T) = -\frac{1}{2}(4)(3) = -6\,.$$

(c) If D is the unit disk then Green's Theorem yields

$$\int_{\partial D} \vec{F}\cdot d\vec{r} = \int_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \int_D y - 3\;dA = \int_D y\;dA - 3\int_D dy = 3\pi\,.$$

Indeed, the function $f(x,y) = y$ is symmetry about the origin [which means $f(-x,-y) = -f(x,y)$] so the integral of $f(x,y)$ over D is zero.

(d) If S is the disk given by $x^2+y^2 \le 1$ and $z = 2$ oriented upwards then $\partial S = C$. Since $\nabla\times\vec{F} = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & x+yz & x^2 \end{bmatrix} = -y\vec{i} + x\vec{j} + \vec{k}$, Stokes' Theorem implies that

$$\int_{\partial S} \vec{F}\cdot d\vec{r} = \int_S (\nabla\times\vec{F})\cdot d\vec{A} = \int_S (-y\vec{i} + 3x\vec{j} + \vec{k})\cdot\vec{k}\;dA = \text{Area}(S) = \pi\,.$$

□

17. Calculate the area of the region within the ellipse $x^2/a^2 + y^2/b^2 = 1$ parameterized by $x = a\cos(t)$, $y = b\sin(t)$ for $0 \le t \le 2\pi$.

Solution. If R is region in the plane and $\vec{F} = x\vec{j}$ then Green's Theorem implies that $\text{Area}(R) = \int_R \, dA = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \, dA = \int_{\partial R} \vec{F} \cdot d\vec{r}$. Using this idea, we can calculate the area of the region enclosed by the given parameterized curves. For the ellipse, we have

$$\begin{aligned}\text{Area} &= \int_{\partial R} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} a\cos(t) b\cos(t) \, dt = ab \int_0^{2\pi} \cos^2(t) \, dt \\ &= ab \left[\frac{1}{2}\cos(t)\sin(t) + \frac{t}{2}\right]_0^{2\pi} = ab\pi \, .\end{aligned}$$

□

18. Compute the flux of the vector field $\vec{F}$ through the surface S.

(a) $\vec{F} = x\vec{i} + y\vec{j}$ and S is the part of the surface $z = 25 - (x^2 + y^2)$ above the disk of radius 5 centered at the origin oriented upward.

(b) $\vec{F} = -y\vec{i} + z\vec{k}$ and S is the part of the surface $z = y^2 + 5$ over the rectangle $-2 \le x \le 1$, $0 \le y \le 1$ oriented upward.

(c) $\vec{F} = y\vec{i} + \vec{j} - xz\vec{k}$ and S is the surface $y = x^2 + z^2$ with $x^2 + z^2 \le 1$ oriented in the positive y-direction.

(d) $\vec{F} = x^2\vec{i} + (y - 2xy)\vec{j} + 10z\vec{k}$ and S is the sphere of radius 5 centered at the origin oriented outward.

(e) $\vec{F} = -z\vec{i} + x\vec{k}$ and S is a square pyramid with height 3 and base on the xy-plane of side length 1.

(f) $\vec{F} = y\vec{j}$ and S is a closed vertical cylinder of height 2 with its base a circle of radius 1 on the xy-plane centered at the origin.

Solution.

(a) The surface S is given by $\vec{r}(s,t) = s\cos(t)\vec{i} + s\sin(t)\vec{j} + (25 - s^2)\vec{k}$ for $0 \le s \le 5$, $0 \le t \le 2\pi$. Since

$$\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(t) & \sin(t) & -2s \\ -s\sin(t) & s\cos(t) & 0 \end{bmatrix} = 2s^2\cos(t)\vec{i} + 2s^2\sin(t)\vec{j} + s\vec{k} \, ,$$

we have

$$\int_S \vec{F}\cdot d\vec{A} = \int_0^{2\pi}\int_0^5 \vec{F}\big(\vec{r}(s,t)\big)\cdot\left(\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\right)\,ds\,dt$$
$$= \int_0^{2\pi}\int_0^5 \big(s\cos(t)\vec{i}+s\sin(t)\vec{j}\big)\cdot\big(2s^2\cos(t)\vec{i}+2s^2\sin(t)\vec{j}+s\vec{k}\big)\,ds\,dt$$
$$= \int_0^{2\pi}\int_0^5 2s^3\cos^2(t)+2s^3\sin^2(t)\,ds\,dt$$
$$= \int_0^{2\pi}\int_0^5 2s^3\,ds\,dt = 2\pi\left[\frac{s^4}{2}\right]_0^5 = 725\pi\,.$$

(b) The surface S is given by $\vec{r}(s,t) = s\vec{i}+t\vec{j}+(t^2+5)\vec{k}$ for $-2\le s\le 1$, $0\le t\le 1$. Since $\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t} = \det\left[\begin{smallmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & 0\\ 0 & 1 & 2t\end{smallmatrix}\right] = -2t\vec{j}+\vec{k}$, we have

$$\int_S \vec{F}\cdot d\vec{A} = \int_{-2}^1\int_0^1 \vec{F}\big(\vec{r}(s,t)\big)\cdot\left(\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\right)\,ds\,dt$$
$$= \int_{-2}^1\int_0^1 \big(-t\vec{i}+(t^2+5)\vec{k}\big)\cdot(-2t\vec{j}+\vec{k})\,dt\,ds$$
$$= \int_{-2}^1\int_0^1 t^2+5\,dt\,ds$$
$$= \left(\int_{-2}^1 ds\right)\left(\int_0^1 t^2+5\,dt\right) = 3\left[\tfrac{1}{3}t^3+5t\right]_0^1 = 16\,.$$

(c) The surface S is given by $\vec{r}(s,t) = s\cos(t)\vec{i}+s^2\vec{j}+s\sin(t)\vec{k}$ for $0\le s\le 1$, $0\le t\le 2\pi$. Since

$$\frac{\partial\vec{r}}{\partial t}\times\frac{\partial\vec{r}}{\partial s} = \det\begin{bmatrix}\vec{i} & \vec{j} & \vec{k}\\ -s\sin(t) & 0 & s\cos(t)\\ \cos(t) & 2s & \sin(t)\end{bmatrix} = -2s^2\cos(t)\vec{i}+s\vec{j}-2s^2\sin(t)\vec{k}$$

we have

$$\int_S \vec{F}\cdot d\vec{A} = \int_0^1\int_0^{2\pi} \vec{F}\big(\vec{r}(s,t)\big)\cdot\left(\frac{\partial\vec{r}}{\partial t}\times\frac{\partial\vec{r}}{\partial s}\right)\,ds\,dt$$
$$= \int_0^1\int_0^{2\pi} \big(s^2\vec{i}+\vec{j}-s^2\sin(t)\cos(t)\vec{k}\big)\cdot\big(-2s^2\cos(t)\vec{i}+s\vec{j}-2s^2\sin(t)\vec{k}\big)\,dt\,ds$$
$$= \int_0^1\int_0^{2\pi} -2s^4\cos(t)+s+2s^4\sin^2(t)\cos(t)\,dt\,ds$$
$$= \int_0^1\left[-2s^4\sin(t)+st+\frac{2}{3}s^4\sin^3(t)\right]_0^{2\pi}\,ds = \int_0^1 2\pi s\,ds = \pi\,.$$

(d) If D is the ball of radius 5 centered at the origin then $\partial D = S$. The Divergence Theorem implies that

$$\int_{\partial D} \vec{F} \cdot d\vec{A} = \int_D (\nabla \cdot \vec{F})\, dV = \int_D 2x + (1 - 2x) + 10\, dV = 11 \cdot \text{Volume}(B) = \frac{5500\pi}{3}\,.$$

(e) If P is the solid square pyramid with height 3 and base on the xy-plane of side length 1 then $\partial P = S$. Hence, the Divergence Theorem gives:

$$\int_{\partial P} \vec{F} \cdot d\vec{A} = \int_P (\nabla \cdot \vec{F})\, dV = \int_P 0\, dV = 0\,.$$

(f) If W is the solid vertical cylinder of height 2 with its base a circle of radius 1 on the xy-plane centered at the origin then $\partial W = S$. Applying the Divergence Theorem yields

$$\int_{\partial W} \vec{F} \cdot d\vec{A} = \int_W (\nabla \cdot \vec{F})\, dV = \int_W 1\, dV = \text{Volume}(W) = 2\pi\,. \qquad \square$$

19. Let $\vec{F} = (8yz - z)\vec{j} + (3 - 4z^2)\vec{k}$.

(a) Show that $\vec{G} = 4yz^2\vec{i} + 3x\vec{j} + xz\vec{k}$ is a vector potential for $\vec{F}$.

(b) Evaluate $\int_S \vec{F} \cdot d\vec{A}$ where S is the upper hemisphere of radius 5 centered at the origin oriented upwards.

Solution.

(a) Since $\nabla \times \vec{G} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4yz^2 & 3x & xz \end{bmatrix} = (8yz - z)\vec{j} + (3 - 4z^2)\vec{k} = \vec{F}$, we see that $\vec{G}$ is a vector potential for $\vec{F}$.

(b) If C is the circle of radius 5 in the xy-plane centered at the origin oriented counterclockwise when viewed from above, then $\partial S = C$. Hence, Stoke's Theorem implies that $\int_S \vec{F} \cdot d\vec{A} = \int_S (\nabla \times \vec{G}) \cdot d\vec{A} = \int_C \vec{G} \cdot d\vec{r}$. Since C can be parameterized by $\vec{r}(t) = 5\cos(t)\vec{i} + 5\sin(t)\vec{j}$ for $0 \le t \le 2\pi$, we have

$$\begin{aligned} \int_S \vec{F} \cdot d\vec{A} = \int_C \vec{G} \cdot d\vec{r} &= \int_0^{2\pi} \vec{G}\big(\vec{r}(t)\big) \cdot \vec{r}\,'(t)\, dt \\ &= \int_0^{2\pi} \big(15\cos(t)\vec{j}\big) \cdot \big(-5\sin(t)\vec{i} + 5\cos(t)\vec{j}\big)\, dt \\ &= 75 \int_0^{2\pi} \cos^2(t)\, dt = 75 \left[\frac{1}{2}\cos(t)\sin(t) + \frac{t}{2}\right]_0^{2\pi} = 75\pi\,. \end{aligned}$$

Alternatively, let D be the disk of radius 5 in the xy-plane centered at the origin oriented upwards. Hence, $\partial D = C = \partial S$ and Stoke's Theorem implies that

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_S (\nabla\times\vec{G})\cdot d\vec{A} = \int_C \vec{G}\cdot d\vec{r} = \int_D (\nabla\times\vec{G})\cdot d\vec{A} \\ &= \int_D \vec{F}\cdot d\vec{A} = \int_D \vec{F}\cdot\vec{k}\,dA = \int_D (3-4(0)^2)\,dA = 3\cdot\text{Area}(D) = 75\pi\,. \quad \square\end{aligned}$$

20. For constants a, b, c and m consider the vector field

$$\vec{F} = (ax+by+5z)\vec{i} + (x+cz)\vec{j} + (3y+mx)\vec{k}\,.$$

(a) Suppose that the flux of $\vec{F}$ through any closed surface is 0. What does this tell you about the values of the constants a, b, c, m?

(b) Suppose instead that the circulation of $\vec{F}$ around any closed curve is 0. What does this tell you about the values of the constants a, b, c, m?

Solution.

(a) Let W be any solid region in 3-space. Since the flux of $\vec{F}$ through any closed surface is 0 the Divergence Theorem implies that

$$0 = \int_{\partial W} \vec{F}\cdot d\vec{A} = \int_W \nabla\cdot\vec{F}\,dV\,.$$

By choosing W to be a sequence of balls centered at (x,y,z) that contract down to (x,y,z) in the limit, we deduce that $\nabla\cdot\vec{F}=0$ everywhere. However, $\nabla\cdot\vec{F}=a$ which means $a=0$.

(b) Let S be any smooth oriented surface with piecewise smooth boundary. Since the circulation of $\vec{F}$ around any closed curve is 0, Stokes' Theorem implies that $0=\int_{\partial S}\vec{F}\cdot d\vec{r} = \int_S(\nabla\times\vec{F})\cdot d\vec{A}$. By choosing S be to a small disk perpendicular to the curl of $\vec{F}$ at (x,y,z), we deduce that $\nabla\times\vec{F}=\vec{0}$. However, we have

$$\nabla\times\vec{F} = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax+by+5z & x+cz & 3y+mx \end{bmatrix} = (3-c)\vec{i} + (5-m)\vec{j} + (1-b)\vec{k}$$

which means that $b=1$, $c=3$ and $m=5$. $\square$