Solution for Review Problems I

MATH 2850 – 004

The detail of the information about the first midterm can be found at http://www.math.utoledo.edu/∼mtsui/calc06sp/exam/midterm1.html

The first midterm will cover 13.1, 13.2, 13.3, 13.4 and 13.5.

The first midterm will be held on Feb. 1 (Wednesday) in class.

My office hours next week(exam week) is Monday 1-3 p.m., Tuesday 12-1:45 p.m. and Wednesday 1-3 p.m.

(1) Let $\overrightarrow{F_3}$ denote third force applied to the object. Since object is to remain stationary, the total force on it must be zero: $\overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} = \overrightarrow{0}$. Hence, we have $\overrightarrow{F_3} = -\overrightarrow{F_1} - \overrightarrow{F_2} = -11\,\overrightarrow{i} + 4\,\overrightarrow{j}$.

(2) (a) We want to find r and s such that $< 2, 1 >= r < 1, 2 > +s < 1, -1 >$. Since $r < 1, 2 > +s < 1, -1 >=< r + s, 2r - s >$, we have $r + s = 2$ and $2r - s = 1$ by comparison the coefficient in each component of the vector. It follows that $r = 1$ and $s = 1$.

(b) The set of vectors are those vectors whose starting points are origin and whose end points lie in the parallelogram spanned by $\{-2\overrightarrow{u} + \overrightarrow{v}, -2\overrightarrow{u} + \overrightarrow{2v}, -\overrightarrow{u} + \overrightarrow{v}, -\overrightarrow{u} + 2\overrightarrow{v}\}$.

(c) The set of vectors are those vectors whose starting points are origin and whose end points lie on the line between $2\overrightarrow{u}$ and $2\overrightarrow{u} + \frac{2}{3}\overrightarrow{v}$.

(3) (a) Using $A = (2, 2, 2)$, $B = (4, 2, 1)$ and $C = (2, 3, 1)$, We have $\overrightarrow{AB} =< 2, 0, -1 >$, $\overrightarrow{AC} =< 0, 1, -1 >$ and $\overrightarrow{BC} =< -2, 1, 0 >$.

The length of the sides of S are:

$||\overrightarrow{AB}|| = \sqrt{2^2 + (0^2 + (-1)^2} = \sqrt{5}$,

$||\overrightarrow{AC}|| = \sqrt{(0)^2 + (1)^2 + (-1)^2} = \sqrt{2}$,

$||\overrightarrow{BC}|| = \sqrt{(-2)^2 + (1)^2 + (0)^2} = \sqrt{5}$.

(b) Let θ be the angle BAC at vertex A.

We have $\cos(\theta) = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{||\overrightarrow{AB}|| \cdot ||\overrightarrow{AC}||} = \frac{<2,0,-1> \cdot <0,1,-1>}{\sqrt{5} \cdot \sqrt{2}} = \frac{1}{\sqrt{10}}$.

(c) First we find

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 2 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} \overrightarrow{i} - \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} \overrightarrow{j} + \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \overrightarrow{k} = \overrightarrow{i} + 2\overrightarrow{j} + 2\overrightarrow{k} =< 1, 2, 2 >$$

Thus the area of the triangle ABC = $\frac{1}{2}||\overrightarrow{AB} \times \overrightarrow{AC}|| = \frac{1}{2}\sqrt{2^2+2^2+1} = \frac{3}{2}$.

(d) The vector $\overrightarrow{AB} \times \overrightarrow{AC} =< 1, 2, 2 >$ is perpendicular to the plane that contains the points A, B and C.

(e) Using $D = (3, 1, 1)$, we have $\overrightarrow{AD} =< 1, -1, -1 >$, Thus $(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} =< 1, 2, 2 > \cdot < 1, -1, -1 >= -2 \neq 0$. Hence A, B , C and D are not on the same plane.

(f) Using $E = (6, 3, -1)$, we have $\overrightarrow{AE} =< 4, -1, -3 >$, Thus $(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AE} =< 1, 2, 2 > \cdot < 4, -1, -3 >= 0$. Hence A, B , C and D are on the same plane.

(g) The volume of the parallelepiped formed by $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$
$= |(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD}| = |-2| = 2$.

(4) Since the geometric definition of the cross product implies that $\|\overrightarrow{v} \times \overrightarrow{w}\| = \|\overrightarrow{v}\|\|\overrightarrow{w}\| \sin(\theta)$ and the geometric definition of the dot product states that $\overrightarrow{v} \cdot \overrightarrow{w} = \|\overrightarrow{v}\|\|\overrightarrow{w}\| \cos(\theta)$, we have

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\|\overrightarrow{v} \times \overrightarrow{w}\|}{\overrightarrow{v} \cdot \overrightarrow{w}} = \frac{\sqrt{(2)^2 + (-3)^3 + (5)^2}}{3} = \frac{\sqrt{38}}{3}.$$

(5) Show that the vectors $(\overrightarrow{b} \cdot \overrightarrow{c})\overrightarrow{a} - (\overrightarrow{a} \cdot \overrightarrow{c})\overrightarrow{b}$ and $\overrightarrow{c}$ are perpendicular.

Since

$$\begin{aligned}((\overrightarrow{b} \cdot \overrightarrow{c})\overrightarrow{a} - (\overrightarrow{a} \cdot \overrightarrow{c})\overrightarrow{b}) \cdot \overrightarrow{c} &= (\overrightarrow{b} \cdot \overrightarrow{c})(\overrightarrow{a} \cdot \overrightarrow{c}) - (\overrightarrow{a} \cdot \overrightarrow{c})(\overrightarrow{b} \cdot \overrightarrow{c}) \\ &= (\overrightarrow{b} \cdot \overrightarrow{c})(\overrightarrow{a} \cdot \overrightarrow{c}) - (\overrightarrow{b} \cdot \overrightarrow{c})(\overrightarrow{a} \cdot \overrightarrow{c}) = 0,\end{aligned}$$

we conclude that $(\overrightarrow{b} \cdot \overrightarrow{c})\overrightarrow{a} - (\overrightarrow{a} \cdot \overrightarrow{c})\overrightarrow{b})$ and $\overrightarrow{c}$ are perpendicular to each other.

(6) The projection formula implies that the component of the wind parallel to the dash is
$Proj_{\overrightarrow{v}}\overrightarrow{w} = \frac{\overrightarrow{w} \cdot \overrightarrow{v}}{||\overrightarrow{v}||^2}\overrightarrow{v} = \frac{16}{40}(2\overrightarrow{i} + 6\overrightarrow{j}) = \frac{4}{5}\overrightarrow{i} + \frac{12}{5}\overrightarrow{j}$ and its magnitude is $||Proj_{\overrightarrow{v}}\overrightarrow{w}|| = \sqrt{(\frac{4}{5})^2 + (\frac{12}{5})^2} = \frac{\sqrt{160}}{5}$. Since $\frac{\sqrt{160}}{5} < 5$, we have $||Proj_{\overrightarrow{v}}\overrightarrow{w}|| < 5$. Therefore, the wind speed measured in the direction of the dash is less than 5 km h^{-1} and the race results will not be disqualified due to an illegal wind.

(7) (a) $\theta = \frac{\pi}{2}$.
(b) $0 \leq \theta < \frac{\pi}{2}$.
(c) $\frac{\pi}{2} < \theta \leq \pi$.
(d) $\theta = 0$ or $\theta = \pi$.

(8) (a) Let $P = (2, 4, 1)$ and $Q = (4, 5, 3)$. The direction of the line is $\overrightarrow{PQ} =< 4 - 2, 5 - 4, 3 - 1 >=< 2, 1, 2 >$. The vector equation is $< x, y, z >=< 2, 4, 1 > +t < 2, 1, 2 >$ or $x = 2 + 2t$, $y = 4 + t$ and $z = 1 + 2t$.

(b) (i) The direction of the line determined by $(13, 14, 12)$ and $(12, 12, 11)$ is $< -1, 2, 1 >$ which is not parallel to $< 2, 1, 2 >$.

(ii) The direction of the line determined by $(7, 8, 6)$ and $(8, 8, 7)$ is $< 1, 0, 1 >$ which is not parallel to $< 2, 1, 2 >$.

(iii) The direction of the line determined by $(9, 10, 8)$ and $(13, 12, 12)$ is $< 4, 2, 4 >= 2 < 2, 1, 2 >$ which is parallel to $< 2, 1, 2 >$.

(c) The direction of the line is $\overrightarrow{PQ} =< 4 - 2, 5 - 4, 3 - 1 >=< 2, 1, 2 >$. The starting point of the line is $(1, 1, 1)$. So the equation of the line is $x = 1 + 2t$, $y = 1 + t$ and $z = 1 + 2t$.

(d) The direction of the line $\frac{x-2}{2} = -\frac{y}{1} = \frac{z-2}{2}$ is $< 2, -1, 2 >$. The starting point of the line is $(1, 1, 1)$. So the equation of the line is $x = 1 + 2t$, $y = 1 - t$ and $z = 1 + 2t$.

(9) (a) (i) Method 1. Since the graph of the linear function intersects the xz-plane along the line $z = 2x-3$, we know it has the form $2x+cy-z-3 = 0$ for some constant c. Moreover, the point $(1, 2, 1)$ lies on the plane if and only $2 + c(2) - 1 - 3 = 0$ which means $c = 1$. Therefore, the required equation is $2x + y - z - 3 = 0$.

(ii) Method 2. We can find two points on the $x - z$ plane with $z = 2x - 3$. Choosing $x = 0$ and $x = 1$, we get $(0, 0, -3)$ and $(1, 0, -1)$. Thus $P = (1, 2, 1)$, $Q = (0, 0, -3)$ and $R = (1, 0, -1)$ determine a plane. A normal vector is $\overrightarrow{n} = \overrightarrow{PQ} \times \overrightarrow{PR} =< -1, -2, -4 > \times < 0, -2, -2 >$

$$= \left\langle \left| \begin{matrix} -2 & -4 \\ -2 & -2 \end{matrix} \right|, -\left| \begin{matrix} -1 & -4 \\ 0 & -2 \end{matrix} \right|, \left| \begin{matrix} -1 & -2 \\ 0 & -2 \end{matrix} \right| \right\rangle =< -4, -2, 2 > .$$

The equation of the plane is determined by a point $P = (1, 2, 1)$ and the normal vector $< -4, -2, 2 >$. This gives $< x - 1, y - 2, z - 1 > \cdot < -4, -2, 2 >= 0$. We get $-4x-2y+2z+4+4-2 = 0$ which can be simplified as $2x + y - z - 3 = 0$.

(b) The equation of the plane with normal vector $\overrightarrow{i} - \overrightarrow{j} + \overrightarrow{k}$ and passing through the point $(1, 1, 1)$ is
$\langle 1, -1, 1\rangle \cdot \langle x - 1, y - 1, z - 1\rangle = (x - 1) - (y - 1) + (z - 1) = 0$ or $x - y + z = 1$.

(c) Let θ be the angle between the planes defined by $3x + 2y - z = 7$ and $x - 4y + 2z = 0$.
We have $\cos(\theta) = \frac{\langle 3,2,-1\rangle \cdot \langle 1,-4,2\rangle}{||\langle 3,2,-1\rangle|| ||\langle 1,-4,2\rangle||} = \frac{-7}{\sqrt{14}\sqrt{21}} = \frac{-1}{\sqrt{6}}$ and $\theta = \arccos(\frac{-1}{\sqrt{6}})$.

(d) The plane that is perpendicular to the planes $3x + 2y - z = 7$ and $x - 4y + 2z = 0$ has normal vector
$\langle 3, 2, -1\rangle \times \langle 1, -4, 2\rangle = \langle 0, -7, -14\rangle$. Thus the equation of the plane is $-7y - 14z = -21$, that is $y + 2z = 3$.

(10) (a) The distance between the point $(1, 2, 3)$ and the plane $2x - 2y + z = 7$ is $\frac{|2 \cdot 1 - 2 \cdot (2) + 3 - 7|}{\sqrt{2^2 + (-2)^2 + (1)^2}} = \frac{|-6|}{3} = 2$.

(b) The plane $4x - 2y + 4z = 7$ can be rewritten as $2x - y + 2z = \frac{7}{2}$. Using the distance formula between planes, the distance between P_1 and P_2 is $\frac{|10 - \frac{7}{2}|}{\sqrt{2^2 + 2^2 + (-1)^2}} = \frac{13}{6}$.

(c) A plane P is drawn through the points $A = (1, -1, 0)$, $B = (0, 1, -1)$ and $C = (1, 0, -1)$. Find the distance between the plane P and the point $(1, 1, 1)$.

The vectors

$\overrightarrow{AB} = (0 - 1)\overrightarrow{i} + (1 + 1)\overrightarrow{j} + (-1 - 0)\overrightarrow{k} = -\overrightarrow{i} + 2\overrightarrow{j} - \overrightarrow{k}$ and

$\overrightarrow{AC} = (1 - 1)\overrightarrow{i} + (0 + 1)\overrightarrow{j} + (-1 - 0)\overrightarrow{k} = \overrightarrow{j} - \overrightarrow{k}$

both lie in the plane. Hence, the vector

$$\overrightarrow{AB} \times \overrightarrow{AC} = \det \begin{bmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -1 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix} = -\overrightarrow{i} - \overrightarrow{j} - \overrightarrow{k}$$

is perpendicular to the plane and the equation of the plane is

$$-1(x - 1) - 1(y + 1) - 1(z - 0) = 0 \quad \text{or} \quad x + y + z = 0\,.$$

The distance between the plane P and the point $(1, 1, 1)$ is $\sqrt{3}$.

(11) Section 12.7 will not be in the test.

(12) The sphere $(x - 2)^2 + (y - 1)^2 + (z + 1)^2 = 4$ has its center at $C = (2, 1, -1)$ with radius 2. Thus the distance between the center and the plane P_1 :

$2x+2y-z=1$ is $\frac{|2\cdot 2+2\cdot 1-(-1)-1|}{\sqrt{2^2+2^2+(-1)^2}} = \frac{6}{3} = 2$ which equals to the radius of the sphere. Thus the plane P_1 intersects the sphere at one point.

The distance between the center and the plane $P_2 : 2x+2y-z=-1$ is $\frac{|2\cdot 2+2\cdot 1-(-1)-(-1)|}{\sqrt{2^2+2^2+(-1)^2}} = \frac{8}{3}$ which is greater than the radius of the sphere. Thus the plane P_2 doesn't intersect the sphere.

The distance between the center and the plane $P_3 : 2x-y+2z=4$ is $\frac{|2\cdot 2+2\cdot 1-(-1)-(4)|}{\sqrt{2^2+2^2+(-1)^2}} = \frac{3}{3} = 1$ which is smaller than the radius of the sphere. Thus the intersection of the plane P_3 with the plane is a circle with radius $\sqrt{3^2-1^2} = \sqrt{8}$.