Solution to Review Problems for Midterm II

MATH 2850 – 004

(1) (a) Change each of the following points from rectangular coordinates to cylindrical coordinates and spherical coordinates:

$$(2,-1,2),(2,-2,-3)\,.$$

(b) Convert the equation $\cos(\phi)=\sin(\theta)$ into rectangular coordinates.

(c) Convert the equation $r\cos(\theta)=z$ into rectangular coordinates.

Solution. (a) $(2,-1,-2)$: Since $r=\sqrt{x^2+y^2}=\sqrt{2^2+(-1)^2}=\sqrt{5}$ and $\tan\theta=\frac{y}{x}=\frac{-1}{2}$, the point in cylindrical coordinates is $\left(\sqrt{5},2\pi-\arctan\left(\frac{1}{2}\right),-2\right)$. Similarly,

$$\rho=\sqrt{x^2+y^2+z^2}=\sqrt{2^2+(-1)^2+(-2)^2}=3\,,$$

$\tan\theta=\frac{y}{x}=-\frac{1}{2}$ and $\cos\phi=\frac{z}{\rho}=\frac{2}{3}$ so the point in spherical coordinates is $\left(3,2\pi-\arctan\left(\frac{1}{2}\right),\arccos\left(\frac{2}{3}\right)\right)$.

$(2,-2,-3)$: Since $r=\sqrt{x^2+y^2}=\sqrt{2^2+(-2)^2}=\sqrt{8}$ and $\tan\theta=\frac{y}{x}=\frac{-2}{2}=-1$, the point in cylindrical coordinates is $\left(\sqrt{5},2\pi-\arctan(1),-3\right)=\left(\sqrt{5},\frac{7}{4}\pi,-3\right)$. Similarly,

$$\rho=\sqrt{x^2+y^2+z^2}=\sqrt{2^2+(-2)^2+(3)^2}=\sqrt{17}\,,$$

$\tan\theta=\frac{y}{x}=-1$ and $\cos\phi=\frac{z}{\rho}=\frac{-2}{\sqrt{17}}$ so the point in spherical coordinates is $\left(\sqrt{17},\frac{7}{4}\pi,\pi-\arccos\frac{2}{\sqrt{17}}\right)$.

(b) Since $\cos(\phi)=\frac{z}{\rho}=\frac{z}{\sqrt{x^2+y^2+z^2}}$, $\sin(\theta)=\frac{y}{\rho\sin(\phi)}$ $=\frac{y}{\sqrt{x^2+y^2+z^2}\sqrt{1-\cos(\phi)^2}}=\frac{y}{\sqrt{x^2+y^2+z^2}\sqrt{1-\frac{z^2}{x^2+y^2+z^2}}}=\frac{y}{\sqrt{x^2+y^2}}$. The equation $\cos(\phi)=\sin(\theta)$ is $\frac{z}{\sqrt{x^2+y^2+z^2}}=\frac{y}{\sqrt{x^2+y^2}}$ in rectangular coordinates.

(c) Since $y=r\cos(\theta)$, the equation $r\cos(\theta)=z$ is $y=z$ in rectangular coordinates.

□

(2) Find the arc-length of the curve $r(t)=\langle\sqrt{2}t,e^t,e^{-t}\rangle$ when $0\le t\le\ln(2)$. (There is a typo in the original problem. $r(t)$ should be $\langle\sqrt{2}t,e^t,e^{-t}\rangle$.)

Solution. Given $r(t)=\langle\sqrt{2}t,e^t,e^{-t}\rangle$, we have $r'(t)=\langle\sqrt{2},e^t,-e^{-t}\rangle$ and $|r'(t)|=\sqrt{2+e^{-2t}+e^{2t}}=\sqrt{(e^{-t}+e^t)^2}=e^{-t}+e^t$. Hence the arc-length of the curve $r(t)=\langle\sqrt{2}t,e^t,e^{-t}\rangle$ between $0\le t\le\ln(2)$ is $\int_0^{\ln(2)}|r'(t)|dt=\int_0^{\ln(2)}(e^{-t}+e^t)dt=-e^{-t}+e^t\big|_0^{\ln(2)}=-e^{-\ln(2)}+e^{\ln(2)}-(-1+1)=-\frac{1}{2}+2=\frac{3}{2}$. Note that $e^{-\ln(2)}=\frac{1}{e^{\ln(2)}}=\frac{1}{2}$.

□

(3) (a) Find parametric equations for the tangent line to the curve $r(t) = \langle t^3, t, t^3\rangle$ at the point $(-1, 1, -1)$.
(b) At what point on the curve $r(t) = \langle t^3, t, t^3\rangle$ is the normal plane (this is the plane that is perpendicular to the tangent line) parallel to the plane $24x + 2y + 24z = 3$?

Solution. (a) Note that $r(t) = \langle t^3, t, t^3\rangle$. We have $r(-1) = \langle -1, 1, -1\rangle$. Taking the derivative of $r(t)$, we get $r'(t) = \langle 3t^2, 1, 3t^3\rangle$. Thus the tangent vector at $t = -1$ is $r'(-1) = \langle 3, 1, 3\rangle$. Therefore parametric equations for the tangent line is $x = -1 + 3t$, $y = 1 + t$ and $z = -1 + 3t$.

(b) The tangent vector at any time t is $r'(t) = \langle 3t^2, 1, 3t^3\rangle$. The normal vector of the normal plane is parallel to $r'(t) = \langle 3t^2, 1, 3t^3\rangle$.

The normal vector of $24x + 2y + 24z = 3$ is $\langle 24, 2, 24\rangle$. So $\frac{24}{3t^2} = \frac{2}{1} = \frac{24}{3t^2}$. This implies that $3t^2 = 12$. So $t = \pm 2$. On the curve $r(t) = \langle t^3, t, t^3\rangle$, the normal plane at the points $r(2) = \langle 8, 2, 8\rangle$ and $r(-2) = \langle -8, 2, -8\rangle$ are parallel to the plane $24x + 2y + 24z = 3$.

□

(4) Find the unit tangent, unit normal, binormal vectors and curvature of the curve $r(t) = \langle 4t, \cos(3t), \sin(3t)\rangle$.

Solution. Given $r(t) = \langle 4t, \cos(3t), \sin(3t)\rangle$, we have $r'(t) = \langle 4, -3\sin(3t), 3\cos(3t)\rangle$ and $|r'(t)| = \sqrt{16 + 9\sin^2(3t) + 9\cos^2(3t)} = \sqrt{25} = 5$. So the unit tangent vector is $T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{5}\langle 4, -3\sin(3t), 3\cos(3t)\rangle$.

Now $T'(t) = \frac{1}{5}\langle 0, -9\cos(3t), -9\sin(3t)\rangle$ and $|T'(t)| = \frac{9}{5}$. So the unit normal vector is $N(t) = \frac{T'(t)}{|T'(t)|} = \langle 0, -\cos(3t), -\sin(3t)\rangle$.

The binormal vector is

$$
\begin{aligned}
B(t) = T(t) \times N(t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{4}{5} & \frac{-3\sin(3t)}{5} & \frac{3\cos(3t)}{5} \\ 0 & -\cos(3t) & -\sin(3t) \end{vmatrix} \\
&= \begin{vmatrix} \frac{-3\sin(3t)}{5} & \frac{3\cos(3t)}{5} \\ -\cos(3t) & -\sin(3t) \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{4}{5} & \frac{3\cos(3t)}{5} \\ 0 & -\sin(3t) \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{4}{5} & \frac{-3\sin(3t)}{5} \\ 0 & -\cos(3t) \end{vmatrix} \vec{k} \\
&= \langle \frac{3}{5}, -\frac{4}{5}\sin(3t), -\frac{4}{5}\cos(3t)\rangle.
\end{aligned}
$$

The curvature $k(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{\frac{9}{5}}{5} = \frac{9}{25}$.

□

(5) Find the linear approximation of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at $(1, 2, 2)$ and use it to estimate $\sqrt{(1.1)^2 + (2.1)^2 + (1.9)^2}$.

Solution. The partial derivatives are $f_x(x, y, z) = \frac{x}{\sqrt{x^2+y^2+z^2}}$, $f_y(x, y, z) = \frac{y}{\sqrt{x^2+y^2+z^2}}$, $f_y(x, y, z) = \frac{z}{\sqrt{x^2+y^2+z^2}}$, $f_x(1, 2, 2) = \frac{1}{3}$ and $f_y(1, 2, 2) = \frac{2}{3}$ and $f_z(1, 2, 2) = \frac{2}{3}$.
The linear approximation of $f(x, y, z)$ at $(1, 2, 2)$ is

$$\begin{aligned} L(x, y, z) &= f(1, 2, 2) + f_x(1, 2, 2)(x - 1) + f_y(1, 2, 2)(y - 2) + f_z(1, 2, 2)(z - 2) \\ &= 3 + \frac{1}{3}(x - 1) + \frac{2}{3}(y - 2) + \frac{2}{3}(z - 2). \end{aligned}$$

Thus $L(1.1, 2.1, 1.9) = 3+\frac{1}{3}(1.1-1)+\frac{2}{3}(2.1-2)+\frac{2}{3}(1.9-2) = 3+\frac{.1+.2-.2}{3} = 3+\frac{.1}{3} \approx 3.033$. Hence $\sqrt{(1.1)^2 + (2.1)^2 + (1.9)^2}$ is about 3.033.

□

(6) (a) Find the equation for the plane tangent to the surface $z = 3x^2 - y^2 + 2x$ at $(1, -2, 1)$.

(b) Find the equation for the plane tangent to the surface $x^2+xy^2+xyz = 4$ at $(1, 1, 2)$.

(c) Find the equation for the line normal to the surface $x^2 + xy^2 + xyz = 4$ at $(1, 1, 2)$.

(d) Find the points on the sphere $x^2 + y^2 + z^2 = 1$ where the tangent plane is parallel to the plane $2x + y - 3z = 2$.

(e) Find the points on the sphere $(x + 1)^2 + (y - 1)^2 + z^2 = 1$ where the tangent plane is parallel to the plane $2x + 2y - z = 1$.

Solution. (a) Let $f(x, y) = 3x^2 - y^2 + 2x$. We have $f_x = 6x + 2$, $f_y = -2y$, $f_x(1, -2) = 8$ and $f_y(1, -2) = 4$. The equation of the tangent plane through the point $(1, -2, 1)$ is

$$\begin{aligned} z &= f(1, -2) + f_x(1, -2)(x - 1) + f_y(1, -2)(y + 2) \\ &= 1 + 8(x - 1) + 4(y + 2) = 8x + 4y + 1. \end{aligned}$$

(b) In general, the normal vector for the tangent plane to the level surface of $F(x, y, z) = k$ at the point (a, b, c) is $\nabla F(a, b, c)$.

The surface $x^2+xy^2+xyz = 4$ can be rewritten as $F(x, y, z) = x^2+xy^2+xyz = 4$, $\nabla F(x, y, z) = \langle 2x + y^2 + yz, 2xy + xz, xy\rangle$ and

$\nabla F(1,1,2) = \langle 5,4,1\rangle$ Thus the equation of the tangent plane to the surface $x^2 + xy^2 + xyz = 4$ at the point $(1,1,2)$ is
$\langle 5,4,1\rangle \cdot \langle x-1, y-1, z-2\rangle = 0$ which yields
$5x - 5 + 4y - 4 + z - 2 = 0$. It can be simplified as $5x + 4y + z - 11 = 0$.

(c) The normal line equation at $(1,1,2)$ is $x = 1+5t$, $y = 1+4t$ and $z = 2+t$.

(d) Recall that the equation of the tangent plane at any point (x_0, y_0, z_0) on the sphere $x^2 + y^2 + z^2 = 1$ is the equation $x_0x + y_0y + z_0z = 1$. (Note that the equation of the tangent plane at any point (x_0, y_0, z_0) on the sphere $x^2+y^2+z^2 = R^2$ is the equation $x_0x+y_0y+z_0z = R$.) The plane $x_0x+y_0y+z_0z = 1$ is parallel to the plane $2x + y - 3z = 2$ if their normal vectors are parallel, that is,

$$\frac{x_0}{2} = \frac{y_0}{1} = \frac{z_0}{-3} = c.$$

Hence $x_0 = 2c$, $y_0 = c$ and $z_0 = -3c$. Recall that (x_0, y_0, z_0) ia a point on the sphere $x^2 + y^2 + z^2 = 1$. Thus $x_0^2 + y_0^2 + z_0^2 = 4c^2 + c^2 + 9c^2 = 1$, $14c^2 = 1$ and $c = \pm\frac{1}{\sqrt{14}}$. We have $(x_0, y_0, z_0) = (\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-3}{\sqrt{14}})$ or $(x_0, y_0, z_0) = (-\frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}})$.

(e) Recall that the equation of the tangent plane at any point (x_0, y_0, z_0) on the sphere $(x+1)^2 + (y-1)^2 + z^2 = 1$ is the equation $(x_0+1)(x+1) + (y_0 - 1)(y-1) + z_0z = 1$. The plane $(x_0+1)(x+1) + (y_0-1)(y-1) + z_0z = 1$ is parallel to the plane $2x + 2y - z = 1$ if there normal vectors are parallel, that is,

$$\frac{x_0+1}{2} = \frac{y_0-1}{2} = \frac{z_0}{-1} = c.$$

Hence $x_0 + 1 = 2c$, $y_0 - 1 = 2c$ and $z_0 = -c$. Recall that (x_0, y_0, z_0) is a point on the sphere $(x+1)^2 + (y-1)^2 + z^2 = 1$. Thus $(x_0+1)^2 + (y_0-1)^2 + z_0^2 = 4c^2 + 4c^2 + c^2 = 1$, $9c^2 = 1$ and $c = \pm\frac{1}{3}$. Recall that $x_0 = 2c - 1$, $y_0 = 2c + 1$ and $z_0 = -c$. We have $(x_0, y_0, z_0) = (-\frac{1}{3}, \frac{5}{3}, -\frac{1}{3})$ or $(x_0, y_0, z_0) = (-\frac{5}{3}, \frac{1}{3}, \frac{1}{3})$.

□

(7) Find the domain and first partial derivatives of the following functions.

(a) $f(s,t) = (s^2 + t^2)\sin(s^2 - t^2)$.

(b) $g(x,y) = \frac{2x-3y}{x+2y}$.

(c) $h(x,y) = \ln(\frac{x+y}{x-y})$.

(d) $k(x,t) = \frac{(3x+4t)e^{(x^2-t^2)}}{x^2+t^2}$.

Solution. (a) $f(s,t) = (s^2+t^2)\sin(s^2-t^2)$.
The domain of f is $\{(s,t)|s \in R \text{ and } t \in R\}$ We have

$$f_s = 2s\sin(s^2-t^2) + 2s(s^2+t^2)\cos(s^2-t^2)$$

and

$$f_t = 2t\sin(s^2-t^2) - 2t(s^2+t^2)\cos(s^2-t^2).$$

(b) $g(x,y) = \frac{2x-3y}{x+2y}$.
The domain of g is $\{(x,y)|x+2y \neq 0\}$
We have

$$g_x = \frac{2(x+2y)-(2x-3y)}{(x+2y)^2} = \frac{7y}{(x+2y)^2}$$

and

$$g_y = \frac{-3(x+2y)-2(2x-3y)}{(x+2y)^2} = \frac{-7x}{(x+2y)^2}.$$

(c) $h(x,y) = \ln(\frac{x+y}{x-y})$.
The domain of g is $\{(x,y)|x-y \neq 0 \text{ and } \frac{x+y}{x-y} > 0\}$
Note that $h(x,y) = \ln(\frac{x+y}{x-y}) = \ln(x+y) - \ln(x-y)$. We have

$$h_x = \frac{1}{x+y} - \frac{1}{x-y} = \frac{-2y}{x^2-y^2}$$

and

$$h_y = \frac{1}{x+y} + \frac{1}{x-y} = \frac{2x}{x^2-y^2}.$$

(d) $k(x,t) = \frac{(3x+4t)e^{(x^2-t^2)}}{x^2+t^2}$.
The domain of k is $\{(x,t)|x^2+t^2 \neq 0\}$, that is, $\{(x,t)|(x,t) \neq (0,0)\}$.
Instead of finding its derivative by brutal force, we will use the logarithm differentiation.
Note that $\ln(k(x,t)) = \ln(\frac{(3x+4t)e^{(x^2-t^2)}}{x^2+t^2}) = \ln(3x+4t) + x^2 - t^2 - \ln(x^2+t^2)$. Thus $(\ln(k(x,t)))_x = (\ln(3x+4t) + x^2 - t^2 - \ln(x^2+t^2))_x$,

$$\frac{k(x,t)_x}{k(x,t)} = \frac{3}{3x+4t} + 2x - \frac{2x}{x^2+t^2}$$

and

$$k_x = (\frac{3}{3x+4t} + 2x - \frac{2x}{x^2+t^2})\frac{(3x+4t)e^{(x^2-t^2)}}{x^2+t^2},$$

Similarly,

$$\frac{k_t}{k} = \frac{4}{3x+4t} - 2t - \frac{2t}{x^2+t^2}$$

and

$$k_t = (\frac{4}{3x+4t} - 2t - \frac{2t}{x^2+t^2})\frac{(3x+4t)e^{(x^2-t^2)}}{x^2+t^2}.$$

□

(8) (a) Verify that $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ is a solution of $u_{xx} + u_{yy} + u_{zz} = 0$.

(b) Show that $v(x,t) = f(x+2t) + g(x-2t)$ is a solution of the wave equation $v_{tt} = 4v_{xx}$.

Solution. (a) We have $u = (x^2+y^2+z^2)^{-\frac{1}{2}}$,

$u_x = -x(x^2+y^2+z^2)^{-\frac{3}{2}},\ u_{xx} = -(x^2+y^2+z^2)^{-\frac{3}{2}} - 3x^2(x^2+y^2+z^2)^{-\frac{5}{2}}$.

The expression is symmetric in x, y and z. Hence we have

$$u_{yy} = (x^2+y^2+z^2)^{-\frac{3}{2}} - 3y^2(x^2+y^2+z^2)^{-\frac{5}{2}}$$

and

$$u_{zz} = (x^2+y^2+z^2)^{-\frac{3}{2}} - 3z^2(x^2+y^2+z^2)^{-\frac{5}{2}}.$$

Thus $u_{xx} + u_{yy} + u_{zz} = 3(x^2+y^2+z^2)^{-\frac{3}{2}} - 3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-\frac{5}{2}} = 3(x^2+y^2+z^2)^{-\frac{3}{2}} - 3(x^2+y^2+z^2)^{-\frac{3}{2}} = 0$.

(b) Using $v(x,t) = f(x+2t) + g(x-2t)$ and chain rule, we have
$v_x = f'(x+2t) + g'(x-2t)$, $v_{xx} = f''(x+2t) + g''(x-2t)$,
$v_t = 2f'(x+2t) - 2g'(x-2t)$, $v_{tt} = 4f''(x+2t) + 4g''(x-2t)$.
Thus $v_{tt} - 4v_{xx} = 4f''(x+2t) + 4g''(x-2t) - 4(f''(x+2t) + g''(x-2t)) = 0$.

□

(9) Use implicit differentiation to find z_x and z_y if $xyz = e^{x^2+y^2+z^2}$.

Solution. Assume $z = z(x,y)$, we have $xyz(x,y) = e^{x^2+y^2+(z(x,y))^2}$.
So $(xyz(x,y))_x = (e^{x^2+y^2+(z(x,y))^2})_x$,
$yz(x,y) + xyz_x = e^{x^2+y^2+(z(x,y))^2}(2x + 2zz_x)$,
$xyz_x - 2e^{x^2+y^2+z^2}zz_x = 2e^{x^2+y^2+z^2}x - yz$
and $z_x = \frac{2e^{x^2+y^2+z^2}x - yz}{xy - 2e^{x^2+y^2+z^2}z}$.

Similarly, $(xyz(x,y))_y = (e^{x^2+y^2+(z(x,y))^2})_y$,
$xz(x,y) + xyz_y = e^{x^2+y^2+(z(x,y))^2}(2y + 2zz_y)$,

$xyz_y - 2e^{x^2+y^2+z^2}zz_y = 2e^{x^2+y^2+z^2}y - xz$
and $z_y = \frac{2e^{x^2+y^2+z^2}y - xz}{xy - 2e^{x^2+y^2+z^2}y}$.

□

(10) Suppose that over a certain region of plane the electrical potential is given by $V(x, y) = x^2 - xy + y^2$.
(a) Find $\nabla V(x, y)$.
(b) Find the direction of the greatest decrease in the electrical potential at the point $(1, 1)$. What is the magnitude of the greatest decrease?
(c) Find the direction of the greatest increase in the electrical potential at the point $(1, 1)$. What is the magnitude of the greatest increase?
(d) Find a direction at the point $(1, 1)$ in which the temperature does not increase or decrease.
(e) Find the rate of change of V at $(1, 1)$ in the direction $\langle 3, -4\rangle$.

Solution. (a) We have

$$\nabla V(x, y) = \langle V_x(x, y), V_y(x, y)\rangle = \langle (x^2 - xy + y^2)_x, (x^2 - xy + y^2)_y\rangle = \langle 2x - y, -x + 2y\rangle$$

(b) Since

$$\nabla V(x, y) = \langle 2x - y, -x + 2y\rangle$$

the direction of the greatest decrease in electrical potential is

$$-\nabla V(1, 1) = -\langle 1, 1\rangle$$

and the magnitude is $-\|\nabla V(1, 1)\| = -\sqrt{2}$.

(c) The direction of greatest increase in electrical potential is

$$\nabla V(1, 1) = \langle 1, 1\rangle$$

and the magnitude is $\|\nabla V(1, 1)\| = \sqrt{2}$.

(d) If $\overrightarrow{u}$ is a direction at which the electrical potential does not increase or decrease, then $D_{\overrightarrow{u}}V(1, 1) = \nabla V(1, 1) \cdot \overrightarrow{u} = 0$. This is equivalent to saying that $\overrightarrow{u}$ is perpendicular to $\nabla V(1, 1)$. If $\overrightarrow{u} = u_1\overrightarrow{i} + u_2\overrightarrow{j}$ then we have $0 = \langle 1, 1\rangle \cdot \overrightarrow{u} = u_1 + u_2$. We may choose $\overrightarrow{u} = \langle 1, -1\rangle$. Therefore, the electrical potential does not change in the direction $\langle 1, -1\rangle$.

(e) The unit vector in the direction $\langle 3, -4\rangle$ is $\overrightarrow{u} = \frac{1}{5}\langle 3, -4\rangle$. Thus the rate of change of V at $(1, 1)$ in the direction $\langle 3, -4\rangle$ is

$$\nabla V(1, 1) \cdot \overrightarrow{u} = \langle 1, 1\rangle \cdot \frac{1}{5}\langle 3, -4\rangle = -\frac{1}{5}.$$

$\square$

(11) Find the local maxima, local minima and saddle points of the following functions. Decide if the local maxima or minima is global maxima or minima. Explain.

(a) $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2$

(b) $f(x, y) = x^2 + y^3 - 3xy$

(c) $f(x, y) = xy + \ln(x) + y^2 - 10$, $x > 0$

Solution. (a) We have $\frac{\partial f}{\partial x} = 6xy - 6x$, $\frac{\partial f}{\partial y} = 3x^2 + 3y^2 - 6y$. Thus (x, y) is a stationary point if $6x(y - 1) = 0$, $3(x^2 + y^2 - 2y) = 0$ From the first equation, we have $x = 0$ or $y = 1$. Suppose $x = 0$, we have $y = 0$ or $y = 2$ from the second equation. Suppose $y = 1$, we have $x = 1$ or $x = -1$ from the second equation. Thus the stationary points of f are $(0, 0)$, $(0, 2)$, $(1, 1)$ and $(-1, 1)$.

The second order partial derivatives are $f_{xx} = 6y - 6$, $f_{xy} = f_{yx} = 6x$ and $f_{yy} = 6y - 6$.

Thus the hessian matrix

$$[D^2 f(x, y)]_{2\times 2} = \begin{pmatrix} 6y - 6 & 6x \\ 6x & 6y - 6 \end{pmatrix}$$

At $(0, 0)$, the hessian matrix is

$$[D^2 f(0, 0)]_{2\times 2} = \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}.$$

We have $f_{xx}(0, 0) = -6 < 0$ and $D = f_{xx}(0, 0)f_{yy}(0, 0) - (f_{xy}(0, 0))^2 = 36 > 0$ This implies that $D^2 f(0, 0)$ negative definite. Thus $(0, 0)$ is a local maximizer with local minimum $f(0, 0) = 0$.

At $(0, 2)$, the hessian matrix is

$$[D^2 f(0, 2)]_{2\times 2} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}.$$

We have $f_{xx}(0,2) = 6 > 0$ and $D = f_{xx}(0,2)f_{yy}(0,2) - (f_{xy}(0,2))^2 = 36 > 0$ This implies that $D^2f(0,2)$ positive definite. Thus $(0,2)$ is a local minimizer with local minimum $f(0,2) = -4$.

At $(1,1)$, the hessian matrix is

$$[D^2f(1,1)]_{2\times 2} = \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}.$$

We have $f_{xx}(1,1) = 0$ and $D = f_{xx}(1,1)f_{yy}(1,1) - (f_{xy}(1,1))^2 = -36 < 0$ This implies that $D^2f(1,1)$ is indefinite. Thus $(1,1)$ is a saddle point.

At $(-1,1)$, the hessian matrix is

$$[D^2f(1,1)]_{2\times 2} = \begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix}$$

We have $f_{xx}(-1,1) = 0$ and $D = f_{xx}(-1,1)f_{yy}(-1,1) - (f_{xy}(-1,1))^2 = -36 < 0$. This implies that $D^2f(-1,1)$ is indefinite. Thus $(-1,1)$ is a saddle point.

(b) The system of equations

$$f_x(x,y) = 2x - 3y = 0 \qquad f_y(x,y) = 3y^2 - 3x = 0$$

implies that $x = \frac{3}{2}y$ and $3\big(y^2 - \frac{3}{2}y\big) = 2y\big(y - \frac{3}{2}\big) = 0$. Thus, $(0,0)$ and $(9/4, 3/2)$ are the critical points. We also have $f_{xx} = 2$, $f_{xy} = f_{yx} = -3$ and $f_{yy} = 6y$.

$$(D^2f(x,y)) = \begin{pmatrix} 2 & -3 \\ -3 & 6y \end{pmatrix}$$

Since

$$\begin{aligned} D = f_{xx}f_{yy} - f_{xy}^2 &= (2)(6y) - (-3)^2 = 12y - 9, \\ Det(D^2f(0,0)) &= -9 < 0, \\ Det(D^2f(9/4,3/2)) &= 18 > 0, \end{aligned}$$

the second derivative test establishes that f has a saddle point at $(0,0)$ and a local minimum at $(9/4, 3/2)$. Because $\lim_{y\to-\infty} f(0,y) = \lim_{y\to-\infty} y^3 = -\infty$, we see that $(9/4, 3/2)$ is not a global minimum.

(c)Solving the system of equations

$$f_x(x,y) = y + \tfrac{1}{x} = 0 \qquad f_y(x,y) = x + 2y = 0\,,$$

we see that $x = -2y$ and $y - \frac{1}{2y} = \frac{2y^2-1}{2y} = 0$. Hence, the only critical point in the region $x > 0$ is $(\sqrt{2}, -1/\sqrt{2})$. We also have $f_{xx} = -\frac{1}{x^2}$, $f_{xy} = f_{yx} = 1$ and

$f_{yy} = 2$.

$$(D^2 f(x,y)) = \begin{pmatrix} -\frac{1}{x^2} & 1 \\ 1 & 2 \end{pmatrix}$$

$$D = f_{xx}f_{yy} - f_{xy}^2 = \left(-\tfrac{1}{x^2}\right)(2) - (2)^2 < 0$$

the second derivative indicates that $(\sqrt{2}, -1/\sqrt{2})$ f is a saddle point.

□

(12) Find rigorously the global maximum/minimum and global maximizer/minimizer of the following functions subject to the given constraint.

(a) $f(x,y) = x^2y^2 - 2x - 2y$, $0 \le x \le 1$ and $0 \le y \le 1$.

(b) $f(x,y) = x^2y^2 - 2x - 2y$, $0 \le x$, $0 \le y$ and $x + y \le 1$.

Solution. (a) Let S denote the region $0 \le x \le 1$ and $0 \le y \le 1$. Since $f(x,y) = x^2y^2 - 2x - 2y$, we have $\nabla f(x,y) = (2xy^2 - 2, 2x^2y - 2)$. and hence the critical point is $(1,1)$. In the following, we use the notation ∂E to denote the boundary of a set E. The boundary $\partial S = S_1 \bigcup S_2 \bigcup S_3 \bigcup S_4$ where $S_1 = \{(x,0)|0 \le x \le 1\}$, $S_2 = \{(0,y)|0 \le y \le 1\}$, $S_3 = \{(x,1)|0 \le x \le 1\}$ and $S_4 = \{(1,y)|0 \le y \le 1\}$.

The restriction of f to S_1 is $f(x,0) = -2x$ where $0 \le x \le 1$. Then $f'(x,0) = -2$. Hence there is no stationary point on S_1.

The restriction of f to S_2 is $f(0,y) = x^2 - 2x$ where $0 \le y \le 1$. Then $f'(0,y) = -2$. Hence there is no critical point on S_2.

The restriction of f to S_3 is $f(x,1) = x^2 - 2x - 2$ where $0 \le x \le 1$. Then $f'(x,0) = 2x - 2$. Hence there is no critical point inside S_3($x = 1$ is on the boundary of S_3).

The restriction of f to S_4 is $f(1,y) = y^2 - 2y - 2$ where $0 \le y \le 1$. Then $f'(0,y) = 2y - 2$. Hence there is no critical point inside S_4($y = 1$ is on the boundary of S_4).

Note that $\partial S_1 \bigcup \partial S_2 \bigcup \partial S_3 \bigcup \partial S_4 = \{(0,0),(1,0),(0,1),(1,1)\}$.

From the computation about, we need to compute the following values of f at the following points $\{(0,0),(1,0),(0,1),(1,1)\}$.

We have

$f(1,1) = -3$, $f(1,0) = f(0,1) = -2$, $f(0,0) = 0$. Hence, the maximum is $f(0,0) = 0$ and the minimum is $f(1,1) = -3$.

(b) Let S denote the region $0 \le x$, $0 \le y$ and $x + y \le 1$. Since $f(x,y) = x^2y^2 - 2x - 2y$, we have $\nabla f(x,y) = (2xy^2 - 2, 2x^2y - 2)$. and hence the critical point is $(1,1)$. The boundary $\partial S = S_1 \bigcup S_2 \bigcup S_3$ where $S_1 = \{(x,0)|0 \le x \le 1\}$, $S_2 = \{(0,y)|0 \le y \le 1\}$, $S_3 = \{(x,y)|0 \le x \le 1, x + y = 1\}$.

The restriction of f to S_1 is $f(x,0) = -2x$ where $0 \le x \le 1$. Then $f'(x,0) = -2$. Hence there is no critical point on S_1.

The restriction of f to S_2 is $f(0,y) = x^2 - 2x$ where $0 \le y \le 1$. Then $f'(0,y) = -2$. Hence there is no critical point on S_2.

Note that $x + y = 1$ on S_3. So $y = 1 - x$ on S_3. The restriction of f to S_3 is $f(x, 1-x) = x^2(1-x)^2 - 2x - 2(1-x) = x^2(1-x)^2 - 2$ where $0 \le x \le 1$. Then $f'(x, 1-x) = 2x(1-x)^2 - 2x^2(1-x)$. Hence the critical point on S_3 is determined by $2x(1-x)^2 - 2x^2(1-x) = 0$, i.e. $2x(x^2 - 2x + 1) - 2x^2 + 2x^3 = 2x^3 - 4x^2 + 2x - 2x^2 + 2x^3 = 4x^3 - 6x^2 + 2x = 2x(2x^2 - 3x + 1) = 2x(x-1)(2x-1) = 0$. So $x = 0$, $x = 1$ or $x = \frac{1}{2}$. Note that $y = 1 - x$. We have $(x,y) = (0,1), (1,0)$ or $(\frac{1}{2}, \frac{1}{2})$.

Note that $\partial S_1 \bigcup \partial S_2 \bigcup \partial S_3 \bigcup \partial S_4 = \{(0,0), (1,0), (0,1), (1,1)\}$.

From the computation about, we need to compute the following values of f at the following points $\{(0,0), (1,0), (0,1), (1,1), (\frac{1}{2}, \frac{1}{2})\}$.
We have

$f(1,1) = -3$, $f(1,0) = f(0,1) = -2$, $f(0,0) = 0$ and $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{16} - 1 = \frac{15}{16}$. Hence, the maximum is $f(0,0) = 0$ and the minimum is $f(1,1) = -3$.

□