Solution to Review Problems for Midterm III
MATH 2850 – 004

The detail of the information about the second midterm can be found at http://www.math.utoledo.edu/~mtsui/calc06sp/exam/midterm3.html

You should also review the homework and quiz problems to prepare for the midterm.

This midterm will cover 15.8 Lagrange multipliers, Chapter 16 (except 16.9) and 17.1-17.2.

(1) Use Lagrange multipliers to find the maximum or minimum values of \( f \) subject to the given constraint.
(a) \( f(x, y) = xy, (1 + x^2)(1 + y^2) = 4 \).

Solution. The constraint is \( g(x, y) = (1 + x^2)(1 + y^2) = 4 \) and the function is \( f(x, y) = xy \). We have \( \nabla f(x, y) = (y, x) \) and \( \nabla g(x, y) = (2x(1 + y^2), 2y(1 + x^2)) \). The equation \( \nabla f(x, y) = \lambda \nabla g(x, y) \) is the same as \( y = 2\lambda x(1 + x^2) \) and \( x = 2\lambda y(1 + y^2) \). Thus the optimizer \((x, y)\) satisfy

(1) \( y = 2\lambda x(1 + x^2) \)
(2) \( x = 2\lambda y(1 + y^2) \)
(3) \((1 + x^2)(1 + y^2) = 4 \)

Multiplying the first two equations, we have \( xy = 4\lambda^2 xy(1 + x^2)(1 + y^2) \). The constraint equation \((1 + x^2)(1 + y^2) = 4 \) implies \( xy = 16\lambda^2 xy \). Thus \( xy = 0 \) or \( \lambda = \pm \frac{1}{4} \). But \( xy = 0, y = 2\lambda x(1 + x^2) \) and \( x = 2\lambda y(1 + y^2) \) imply \((x, y) = (0, 0) \) which doesn’t satisfy the constraint equation. Suppose \( \lambda = \frac{1}{4} \), we have \( y = \frac{1}{2} x(1 + x^2) \) and \( x = \frac{1}{2} y(1 + x^2) = \frac{1}{4} x(1 + x^2)^2 \). Thus \( x = 0 \) or \((1 + x^2)^2 = 4 \). This implies \((x, y) = (1, 1), (-1, -1) \). Suppose \( \lambda = -\frac{1}{4} \), we have \( y = -\frac{1}{2} x(1 + x^2) \) and \( x = -\frac{1}{2} y(1 + x^2) = \frac{1}{4} x(1 + x^2)^2 \). Thus \( x = 0 \) or \((1 + x^2)^2 = 4 \). This implies \((x, y) = (1, -1), (-1, 1) \). We have \( f(1, 1) = f(-1, -1) = 1 \) and \( f(1, -1) = f(-1, 1) = -1 \). Thus the maximizers are \((1, 1) \) and \((-1, -1) \) with maximum 1. The minimizers are \((-1, 1) \) and \((1, -1) \) with minimum \(-1 \). 

(b) \( f(x, y, z) = x^2 - y^2, x^2 + y^2 = 2 \)

Solution. Let \( f(x, y) = x^2 - y^2 \) and \( g(x, y) = x^2 + y^2 = 2 \). The necessary conditions for the optimizer \((x, y)\) are

\( \nabla f(x, y) = \lambda \nabla g(x, y) \) and the constraint equations \( x^2 + y^2 = 2 \) which are:
Since $\nabla f(x, y) = (2x, -2y)$ and $\nabla g(x, y) = (2x, 2y)$, thus $(x, y)$ must satisfy

\begin{align*}
(4) & \quad 2x = 2\lambda x \\
(5) & \quad -2y = 2\lambda y \\
(6) & \quad x^2 + y^2 = 2
\end{align*}

From (4), (5), we get $4x^2 + 4y^2 = 4\lambda^2(x^2 + y^2)$. Since $x^2 + y^2 = 2m$ we have $\lambda^2 = 1$. So $\lambda = \pm 1$. If $\lambda = 1$, then eq(4) is always true and we get $y = 0$ by eq(5). Using $x^2 + y^2 = 2$, we get $x = \pm \sqrt{2}$.

If $\lambda = -1$, then eq(5) is always true and we get $x = 0$ by eq(4). Using $x^2 + y^2 = 2$, we get $y = \pm \sqrt{2}$.

So the candidates are $(\sqrt{2}, 0), (-\sqrt{2}, 0), (0, \sqrt{2})$ and $(0, -\sqrt{2})$.

So $f((\sqrt{2}, 0)) = f((-\sqrt{2}, 0)) = 2$ and $f((0, \sqrt{2}, 0)) = f((0, -\sqrt{2}, 0)) = -2$.

Thus the maximum is 2, the minimum is $-2$, the maximizers are $(\sqrt{2}, 0), (-\sqrt{2}, 0)$, and the minimizers are $(0, \sqrt{2}, 0)$ and $(0, -\sqrt{2}, 0)$.

\[ \square \]

(c) $f(x, y, z) = x + y + z, x^2 + y^2 + z^2 = 1.$

**Solution.** Let $f(x, y, z) = x + y + z$ and $g(x, y, z) = x^2 + y^2 + z^2 = 1$.

We have $\nabla f(x, y, z) = (1, 1, 1)$ and $\nabla g(x, y, z) = (2x, 2y, 2z)$.

The necessary conditions for the optimizer $(x, y, z)$ are

$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and the constraint equations which are:

\begin{align*}
(7) & \quad 1 = 2\lambda x \\
(8) & \quad 1 = 2\lambda y \\
(9) & \quad 1 = 2\lambda z \\
(10) & \quad x^2 + y^2 + z^2 = 1
\end{align*}

From (7), (8) (9) and (10), we know that $\lambda \neq 0$, $x = \frac{1}{2\lambda}, y = \frac{1}{2\lambda}$ and $z = \frac{1}{2\lambda}$.

Plugging into (10), we get $\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 1, \frac{3}{4\lambda^2} = 1$ and $\lambda = \pm \sqrt{3}$. So $(x, y, z) = (\frac{1}{2\lambda}, \frac{1}{2\lambda}, \frac{1}{2\lambda}) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ or $(x, y, z) = (\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}})$.

We have $f((\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})) = \frac{3}{\sqrt{3}} = \sqrt{3}$ and $f((\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}})) = -\frac{3}{\sqrt{3}} = -\sqrt{3}$.

Thus the maximizers are $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ with maximum $\sqrt{3}$. The minimizers are $(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}})$ with minimum $-\sqrt{3}$.

\[ \square \]
(2) Using Riemann sums with two subdivisions in each direction, find upper and lower bounds for the volume under the graph of \( f(x, y) = 1 + x^2 + y^2 \) above the rectangle \( R \) with \( 0 \leq x \leq 2, 0 \leq y \leq 2 \).

**Solution.** Let \( f(x, y) = 1 + x^2 + y^2 \). We have \( f_x = 2x \geq 0 \) and \( f_y = 2y \geq 0 \) in the rectangle \( R \). The lower estimate is \( f(0, 0) \cdot 1 + f(1, 0) \cdot 1 + f(0, 1) \cdot 1 + f(1, 1) \cdot 1 = 1 + 2 + 2 + 3 = 8 \).

The upper estimate is \( f(1, 1) \cdot 1 + f(1, 2) \cdot 1 + f(2, 1) \cdot 1 + f(2, 2) \cdot 1 = 3 + 5 + 5 + 9 = 22 \).

(3) Compute the following iterated integrals.

(a) \( \int \int_D \frac{6x}{y^3+1} \, dA \) where \( D = \{(x, y)|0 \leq x \leq y, 0 \leq y \leq 1\} \).

**Solution.** \( \int \int_D \frac{6x}{y^3+1} \, dA = \int_0^1 \int_0^y \frac{6x}{y^3+1} \, dx \, dy = \int_0^1 \frac{3x^2}{y^3+1} \bigg|_0^y \, dy = \int_0^1 \frac{6y^2}{y^3+1} \, dy \). Let \( u = y^3 + 14 \). Then \( du = 3y^2 \, dy \), \( y^2 \, dy = \frac{1}{3} \, du \) and \( \int \frac{3y^2}{y^3+1} \, dy = \int \frac{3}{5u} \, du = \ln|u| + C = \ln|y^3 + 1| + C \). Hence \( \int \int_D \frac{6x}{y^3+1} \, dA = \ln|y^3 + 1| \bigg|_0^1 = \ln(2) \).

(b) \( \int_0^1 \int_0^1 \frac{y^2x^2}{x^2+1} \, dx \, dy \)

Let \( D = \{(x, y)|\sqrt{y} \leq x \leq 1, 0 \leq y \leq 1\} \) Then \( 0 \leq y \leq x^2 \) and \( 0 \leq x \leq 1 \). So \( D \) is the same as \( \{(x, y)|0 \leq x \leq 1, 0 \leq y \leq x^2\} \).

We have \( \int_0^1 \int_0^1 \frac{y^2x^2}{x^2+1} \, dx \, dy = \int_0^1 \int_0^x \frac{y^2x^2}{x^2+1} \, dx \, dy = \int_0^1 \frac{y^2x^2}{2x} \bigg|_0^1 \, dx = \int_0^1 \frac{x^2}{2} \, dx = \frac{\sqrt{x^3}}{3} \bigg|_0^1 = \frac{1}{4} - \frac{1}{4} \).

(c) \( \int_0^1 \int_x^1 \cos(y^2) \, dy \, dx \)

**Solution.** Let \( D = \{(x, y)|0 \leq x \leq 1, x \leq y \leq 1\} \). Since \( x \leq y \) and \( 0 \leq x \), we have \( 0 \leq x \leq y \). Since \( x \leq y \leq 1 \) and \( 0 \leq x \), we have \( 0 \leq y \leq 1 \). So \( D \) is the same as \( \{(x, y)|0 \leq x \leq y, 0 \leq y \leq 1\} \).

We have \( \int_0^1 \int_x^1 \cos(y^2) \, dy \, dx = \int_0^1 \int_0^y \cos(y^2) \, dy \, dx = \int_0^1 x \cos(y^2) \, dx = \int_1^y \sin(y^2) \, dy = \sin(y^2) \bigg|_0^1 = \sin(1) \).

(d) \( \int_{-3}^3 \int_0^{\sqrt{9-y^2}} \sqrt{x^2+y^2} \, dx \, dy \)

**Solution.** The region of integration is \( \{(x, y)|0 \leq x \leq \sqrt{9-y^2}, -3 \leq y \leq 0\} \). The is the region in fourth quadrant. In polar coordinates, it is \( R = \{(r, \theta)|0 \leq r \leq 3, -\frac{\pi}{2} \leq \theta \leq 0\} \). We also have \( \sqrt{x^2+y^2} = (r^2)^{\frac{1}{2}} = r \) and
\[
\int_0^3 \int_0^\sqrt{9-y^2} \sqrt{x^2+y^2} \, dx \, dy = \int_0^\pi \int_0^r r \cdot r \, dr \, d\theta \\
= \int_0^\pi \int_0^3 r^2 \, dr \, d\theta = \int_0^{2\pi} \frac{r^3}{3} \bigg|_0^3 d\theta = 3\theta \bigg|_0^\pi = \frac{3\pi}{2}.
\]

(e) \[
\int_0^2 \int_0^{-\sqrt{4-x^2}} e^{-x^2-y^2} \, dy \, dx
\]

**Solution.** The region of integration is \( \{(x,y) | 0 \leq x \leq 2, -\sqrt{4-x^2} \leq y \leq 0\} \). This is the region in the fourth quadrant. In polar coordinates, it is \( R = \{(r,\theta) : 0 \leq r \leq 3, \frac{\pi}{2} \leq \theta \leq 0\} \). We also have \( x^2 + y^2 = r^2 \) and
\[
\int_0^2 \int_0^{-\sqrt{4-x^2}} e^{-x^2-y^2} \, dy \, dx = \int_0^\pi \int_0^2 e^{-r^2} \cdot r \, dr \, d\theta \\
= \int_0^\pi \left[ -\frac{e^{-r^2}}{2} \right]_0^2 \, d\theta = -\left(\frac{e^{-4}}{2} - \frac{1}{2}\right) \cdot \frac{\pi}{2}.
\]

(f) \[
\int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{\sqrt{x^2+y^2}} xyz \, dz \, dx \, dy
\]

**Solution.** The region of integration is \( \{(x,y,z) | 0 \leq x \leq \sqrt{1-y^2}, 0 \leq y \leq 1, x^2 + y^2 \leq z \leq \sqrt{x^2 + y^2}\} \). In cylindrical coordinates, it is \( R = \{(r,\theta,z) : 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}, r^2 \leq z \leq r\} \). Recall that \( x = r \cos(\theta), \quad x = r \sin(\theta) \) We have \( xyz = r \cos(\theta) \cdot r \sin(\theta) = r^2 \cos(\theta) \cdot \sin(\theta) \) and
\[
\int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{\sqrt{x^2+y^2}} xyz \, dz \, dx \, dy = \int_0^\pi \int_0^1 \int_0^r r^3 \cos(\theta) \cdot \sin(\theta)z \cdot r \, dz \, dr \, d\theta \\
= \int_0^\pi \int_0^1 \int_0^r r^3 z^2 \cos(\theta) \cdot \sin(\theta) \, dz \, dr \, d\theta \\
= \int_0^\pi \int_0^1 \frac{r^3}{2} \cos(\theta) \cdot \sin(\theta) \, d\theta \\
= \int_0^\pi \frac{r^3}{3} \cos(\theta) \cdot \sin(\theta) \, d\theta \\
= \frac{1}{90} \sin^2(\theta) \bigg|_0^\pi = \frac{1}{90}.
\]

(g) \[
\int_{-1}^1 \int_{\sqrt{1-x^2}}^{1-x^2} \int_{x^2+y^2}^{(x^2+y^2)^{\frac{3}{2}}} dz \, dy \, dx
\]

**Solution.** The region of integration is \( \{(x,y,z) | -1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, x^2 + y^2 \leq z \leq 2 - x^2 - y^2\} \). In cylindrical coordinates, it is \( R = \{(r,\theta,z) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, r^2 \leq z \leq 2 - r^2\} \). Recall that \( x = r \cos(\theta), \quad x = r \sin(\theta) \) We have \( (x^2+y^2)^{\frac{3}{2}} = r^3 \) and
\[
\int_{-1}^1 \int_{\sqrt{1-x^2}}^{1-x^2} \int_{x^2+y^2}^{(x^2+y^2)^{\frac{3}{2}}} dz \, dy \, dx = \int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} r^2 \cdot r \, dz \, dr \, d\theta \\
= \int_0^{2\pi} \int_0^1 r^4 \cdot r \, dr \, d\theta \\
= \int_0^{2\pi} r^4 (2-2r^2) \, dr \, d\theta
\]
\[= \int_{0}^{2\pi} \int_{0}^{1} (\frac{2r^5}{5} - \frac{2r^7}{7}) d\theta \]
\[= \int_{0}^{\frac{x}{2}} \frac{4}{35} d\theta \]
\[= \frac{8x}{35}. \]

(h) \[\int_{-2}^{2} \int_{0}^{\sqrt{9-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2 + y^2 + z^2} \, dz \, dx \, dy \]

**Solution.** In spherical coordinates, the region \( E = \{(x, y, z)|0 \leq x \leq \sqrt{4-y^2}, -2 \leq y \leq 2, -\sqrt{4-x^2-y^2} \leq z \leq \sqrt{4-x^2-y^2}\} \) is described by the inequalities \(0 \leq \rho \leq 2, 0 \leq \theta \leq \pi\) and \(0 \leq \phi \leq \pi\). Note that \(y = \rho \sin(\phi) \cos(\theta)\) Hence, the integral is

\[\int_{-2}^{2} \int_{0}^{\sqrt{9-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2 + y^2 + z^2} \, dz \, dx \, dy = \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{\pi/4} \rho^5 \sin^3(\phi) \cos^2(\theta) \, d\rho \, d\theta \, d\phi \]

\[= \left( \int_{0}^{\pi} \cos^2(\theta) \, d\theta \right) \left( \int_{0}^{\pi} \sin^3(\phi) \, d\phi \right) \left( \int_{0}^{\pi/4} \rho^5 \, d\rho \right) \]

\[= \left( \frac{\theta + \sin(\theta)}{2} \right) \left( \frac{1}{3} \cos^3(\phi) \right) \left( \frac{\rho^6}{6} \right) \bigg|_{0}^{\pi/4} \]

\[= \frac{\pi}{2} \cdot \frac{4}{3} \cdot \frac{64}{6} = \frac{64\pi}{9}. \]

(i) \[\int_{0}^{3} \int_{0}^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2 + y^2 + z^2) \, dz \, dx \, dy \]

**Solution.** In spherical coordinates, the region \( E = \{(x, y, z)|0 \leq x \leq \sqrt{9-y^2}, 0 \leq y \leq 3, \sqrt{x^2 + y^2} \leq z \leq \sqrt{18-x^2-y^2}\} \) is described by the inequalities \(0 \leq \rho \leq \sqrt{18}, 0 \leq \theta \leq \frac{\pi}{2}\) and \(0 \leq \phi \leq \frac{\pi}{4}\). Note that \(\sqrt{x^2+y^2} = \sqrt{18-x^2-y^2}\) if \(x^2+y^2 = 9\) and \(z = \sqrt{x^2+y^2}\) is \(\phi = \frac{\pi}{4}\) in spherical coordinates.
(4) Find the volume of the following regions:

(a) The solid bounded by the surface \( z = x\sqrt{x^2 + y} \) and the planes \( x = 0, x = 1, y = 0, y = 1 \) and \( z = 0 \).

**Solution.** The volume is \( \int_0^1 \int_0^1 x\sqrt{x^2 + y} \, dx \, dy \). Let \( u = x^2 + y \). Then \( du = 2x \, dx \) and \( \int x\sqrt{x^2 + y} \, dx = \int \frac{u^{1/2}}{2} \, du = \frac{u^{3/2}}{3} + C = \frac{(x^2 + y)^{3/2}}{3} + C \).

So \( \int_0^1 \int_0^1 x\sqrt{x^2 + y} \, dx \, dy = \int_0^1 \frac{(x^2 + y)^{3/2}}{3} \, dy \)

\[
= \int_0^1 \left( \frac{1+y^{3/2}}{3} \right) - \left( \frac{y^{3/2}}{3} \right) \, dy = \frac{2(1+y)^{5/2}}{15} - \frac{2(y)^{5/2}}{15} \bigg|_0^1 = \frac{2(2)^{5/2}}{15} - \frac{2}{15} - (\frac{2}{15} - 0) \\
= \frac{2(2)^{5/2}}{15} - \frac{2}{15} - \frac{4}{15} = \frac{8\sqrt{2}}{15} - \frac{4}{15} \quad \square
\]

(b) The solid that lies between the sphere \( x^2 + y^2 + z^2 = 4 \), above the \( x - y \) plane, and below the cone \( z = \sqrt{x^2 + y^2} \).

**Solution.** The region is bounded above by the hemisphere \( z = \sqrt{4 - x^2 - y^2} \) and below by the cone \( z = \sqrt{x^2 + y^2} \). We have \( \sqrt{x^2 + y^2} \leq z \leq \sqrt{4 - x^2 - y^2} \). Thus \( x^2 + y^2 \leq z^2 \leq 4 - x^2 - y^2 \) and \( x^2 + y^2 \leq 2 \)

In polar coordinates, this region \( x^2 + y^2 \leq 2 \) is \( R = \{(r,\theta) : 0 \leq r \leq \sqrt{2}, 0 \leq \theta \leq 2\pi \} \). Note that \( \sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2} \) and \( \sqrt{x^2 + y^2} = \sqrt{r^2} = r \).

Hence, we can compute the volume of the region by finding the volume under the graph of \( \sqrt{4 - r^2} \) above the disk \( R = \{(r,\theta) : 0 \leq r \leq \sqrt{2}, 0 \leq \theta \leq 2\pi \} \).

Note that \( x^2 + y^2 = \rho^2 \) Hence, the integral is

\[
\int_0^\pi \int_0^\pi \int_0^{\sqrt{18}} \rho^2 \cdot \rho^2 \sin(\phi) \, d\rho \, d\theta \, d\phi = \int_0^\pi \int_0^\pi \int_0^{\sqrt{18}} \rho^4 \sin(\phi) \, d\rho \, d\theta \, d\phi
\]

\[
= \left( \frac{\pi}{2} \right) \left( -\cos(\phi) \right) \bigg|_0^\pi \left( \frac{\rho^5}{5} \right) \bigg|_0^{\sqrt{18}} = \frac{\pi}{2} \cdot (-1 + 1) \cdot \left( \frac{18^2 \cdot \sqrt{18}}{5} \right) \quad \square
\]
\[ \theta \leq 2\pi \} \text{ and subtracting the volume under the graph of } r \text{ above } R. \]

Therefore, we have

\[
\text{Volume} = \int_{0}^{2\pi} \int_{0}^{\sqrt{2}} (\sqrt{4-r^2}) \, r \, dr \, d\theta - \int_{0}^{2\pi} \int_{0}^{\sqrt{2}0} (r) \, r \, dr \, d\theta
\]

\[= \int_{0}^{2\pi} \int_{0}^{\sqrt{2}0} \left[ -\frac{1}{3}(4-r^2)^{3/2} - \frac{1}{3}r^3 \right] \sqrt{2} \, d\theta\]

\[= \frac{1}{3}(-8\sqrt{2}\pi + 16\pi) . \]

(c) The solid bounded by the plane \( x + y + z = 3 \), \( x = 0 \), \( y = 0 \) and \( z = 0 \).

\[\text{Solution.} \text{ The region } E \text{ bounded by the } xy, yz, xz \text{ planes and the plane } x + y + z = 3 \text{ is the set } \{ (x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 3, 0 \leq y \leq 3-x, 0 \leq z \leq 3-x-y \} . \text{ The volume of } E \text{ is} \]

\[
\int \int \int_{E} dV = \int_{0}^{3} \int_{0}^{3-x} \int_{0}^{3-x-y} dz \, dy \, dx = \int_{0}^{3} \int_{0}^{3-x} z \bigg|_{0}^{3-x-y} dy \, dx
\]

\[= \int_{0}^{3} \int_{0}^{3-x} \left(3-x-y\right) \, dy \, dx = \int_{0}^{3} \left(3y - xy - \frac{y^2}{2}\right) \bigg|_{0}^{3-x} dx \quad \text{(by substitution } u=4-x-2y)\]

\[= \int_{0}^{3} \left(3(3-x) - x(3-x) - \frac{(3-x)^2}{2}\right) dx = \int_{0}^{3} \left(9 - 3x + 3 - 3x + x^2 - \frac{(x^2 - 6x + 9)}{2}\right) dx
\]

\[= \int_{0}^{3} \left(\frac{9}{2} - 3x + \frac{x^2}{2}\right) dx = \frac{9x}{2} - \frac{3x^2}{2} + \frac{x^3}{6} \bigg|_{0}^{3} = \frac{27}{2} - \frac{27}{2} + \frac{27}{6} = \frac{9}{2} . \]

(d) The region bounded by the cylinder \( x^2 + y^2 = 4 \) and the plane \( z = 0 \) and \( y + z = 3 \).

\[\text{Solution.} \text{ The region is bounded above by the plane } z = 3-y \text{ and below by } z = 0 . \text{ In polar coordinates, this region } x^2 + y^2 \leq 4 \text{ is } R = \{ (r, \theta) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi \} . \text{ Note that } z = 3-y = 3-r \cos(\theta) \text{ Hence, we can} \]
compute the volume of the region by

\[
\text{Volume} = \int_0^{2\pi} \int_0^2 (3 - r \cos(\theta)) \ r dr \ d\theta
\]

\[
= \int_0^{2\pi} \int_0^2 3r - r^2 \cos(\theta) \ dr \ d\theta = \int_0^{2\pi} \left[ \frac{3}{2} r^2 - \frac{1}{3} r^3 \cos(\theta) \right]_0^2 \ d\theta
\]

\[
= \int_0^{2\pi} \left[ 6 - \frac{8}{3} \cos(\theta) \right] \ d\theta = 12\pi.
\]

(5) Find the area of the following surfaces.

(a) The cone \( z = \sqrt{x^2 + y^2} \) between the planes \( z = 1 \) and \( z = 2 \)

**Solution.** We have \( z = f(x, y) = \sqrt{x^2 + y^2} \), \( f_x = \frac{x}{\sqrt{x^2 + y^2}} \), \( f_y = \frac{y}{\sqrt{x^2 + y^2}} \) and

\[
\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \left( \frac{x}{\sqrt{x^2 + y^2}} \right)^2 + \left( \frac{y}{\sqrt{x^2 + y^2}} \right)^2} = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} = \sqrt{2}.
\]

Since \( 1 \leq z = \sqrt{x^2 + y^2} \leq 2 \), we have \( 1 \leq x^2 + y^2 \leq 4 \). The region \( E = \{(x, y)|1 \leq x^2 + y^2 \leq 4\} \) is \( \{(r, \theta)|0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\} \) in polar coordinates. Hence the area of surface is

\[
\iint_E \sqrt{1 + f_x^2 + f_y^2} \, dxdy = \int_0^{2\pi} \int_0^2 r \ dr \ d\theta = \int_0^{2\pi} \int_0^2 \sqrt{2} \ r \ dr \ d\theta = 2\sqrt{2} \cdot 2\pi = 4\sqrt{2}\pi.
\]

(b) Given by \( \{(x, y, z)|x^2 + y^2 = 1, 0 \leq z \leq xy, x \geq 0, y \geq 0\} \).

**Solution.** Note that the curve \( x^2+y^2 = 1, x \geq 0 \) and \( y \geq 0 \) can be parameterized by \( x = \cos(t), y = \sin(t) \) with \( 0 \leq t \leq \frac{\pi}{2} \). So \( ds = \sqrt{x'(t)^2 + y'(t)^2} \, dt = \sqrt{\sin^2(t)} \, dt \) and \( z = xy = \cos(t) \sin(t) \). So the area is \( \int_0^2 xy \, ds = \int_0^{\pi/2} \cos(t) \sin(t) \, dt = \frac{\sin^2(t)}{2} \bigg|_0^{\pi/2} = 1/2 \). 

(6) Rewrite the integral \( \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} f(x, y, z) \, dz \, dy \, dx \) as an iterated integral in the order of \( dx \, dy \, dz \)

**Solution.** The region of integration is \( E = \{(x, y, z)|0 \leq z \leq 1 - y, x^2 \leq y \leq 1, -1 \leq x \leq 1\} \). Since \( x^2 \leq y \) and \( -1 \leq x \), we have \( -1 \leq x \leq \sqrt{y} \). Using \( z \leq 1 - y, x^2 \leq y \) and \( -1 \leq x \leq 1 \), we have \( 0 \leq y \leq 1 - z \). Using \( 0 \leq z \leq 1 - y \) and \( 0 \leq y \), we have \( 1 - y \leq 1 \) and \( 0 \leq z \leq 1 \).

So \( \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} f(x, y, z) \, dz \, dy \, dx = \int_0^1 \int_0^{1-z} \int_{-1}^{1} f(x, y, z) \, dx \, dy \, dz \)

(7) Evaluate the following line integrals.
(a) $\int_C 2z\, ds$ where $C$ is given by $x = \cos(t)$, $y = \sin(t)$, $z = t$, $0 \leq \pi$.

\textbf{Solution.} Note that \( \frac{dx}{dt} = \frac{d}{dt}(\cos(t)) = -\sin(t) \), \( \frac{dy}{dt} = \frac{d}{dt}(\sin(t)) = \cos(t) \) and \( \frac{dz}{dt} = \frac{d}{dt}(t) = 1 \).

Since $x = \cos(t)$, $y = \sin(t)$ and $z = t$,
\[
\int_C 2z\, ds = \int_0^{2\pi} 2t \sqrt{(-\sin(t))^2 + (\cos(t))^2} \, dt = \int_0^{2\pi} 2\sqrt{2}t \, dt = 2\sqrt{2} \int_0^{2\pi} t \, dt = 2\sqrt{2} \frac{\pi^2}{2} = 4\sqrt{2}\pi^2.
\]

(b) Evaluate $\int_C F \cdot dr$ where $F =< y, x >$ and $C$ is given by $r(t) = (t, t^2)$ where $0 \leq t \leq 1$.

\textbf{Solution.} Note that $r(t) = (t, t^2)$ with $0 \leq t \leq 1$ is part of the parabola between $(0, 0)$ and $(1, 1)$. Also $r'(t) = (x'(t), y'(t)) = (1, 2t)$.
\[
\int_C F \cdot dr = \int_0^1 (y(t), x(t)) \cdot (x'(t), y'(t)) \, dt = \int_0^1 (t^2, t) \cdot (1, 2t) \, dt = \int_0^1 3t^2 \, dt = t^3\big|_0^1 = 1
\]

(c) $\int_C y\, dx + x\, dy$ where $C$ is the line between $A = (1, 1)$ and $B = (-1, 5)$.

\textbf{Solution.} The equation of the line between $A = (1, 1)$ and $B = (-1, 5)$ is $r(t) = (x(t), y(t)) = (1 - 2t, 1 + 4t)$ with $0 \leq t \leq 1$. Also $r'(t) = (x'(t), y'(t)) = (-2, 4)$.
\[
\int_C y\, dx + x\, dy = \int_0^1 y(t)x'(t) + x(t)y'(t) \, dt = \int_0^1 (1 + 4t) \cdot (-2) + (1 - 2t) \cdot 4 \, dt = \int_0^1 -2 - 8t + 4 - 8t \, dt = \int_0^1 2 - 16t \, dt = 2t - 8t^2\big|_0^1 = -6
\]

(d) $\int_C F \cdot dr$ where $F =< x, y, z >$ and $C$ is given by $r(t) = (\cos(t), \sin(t), t)$ where $0 \leq t \leq \pi$.

\textbf{Solution.} Note that \( \frac{dx}{dt} = \frac{d}{dt}(\cos(t)) = -\sin(t) \), \( \frac{dy}{dt} = \frac{d}{dt}(\sin(t)) = \cos(t) \) and \( \frac{dz}{dt} = \frac{d}{dt}(t) = 1 \).

Since $x = \cos(t)$, $y = \sin(t)$ and $z = t$,
\[
\int_C F \cdot dr = \int_0^\pi (\cos(t), \sin(t), t) \cdot (-\sin(t), \cos(t), 1) \, dt = \int_0^\pi t dt = \frac{\pi^2}{2}
\]

(e) $\int_C F \cdot dr$ where $F =< x, y, z >$ and $C$ is the line between $A = (1, 1, 1)$ and $B = (-1, 0, 3)$.

\textbf{Solution.} The equation of the line between $A = (1, 1, 1)$ and $B = (-1, 0, 3)$ is $r(t) = (x(t), y(t), z(t)) = (1 - 2t, 1 - t, 1 + 2t)$ with $0 \leq t \leq 1$. Also $r'(t) = (x'(t), y'(t), z'(t)) = (-2, -1, 2)$. 

\[ \int_0^1 F \cdot dr = \int_0^1 (1 - 2t, 1 - t, 1 + 2t) \cdot (-2, -1, 2) \, dt = \int_0^1 (-2 + 4t) + (-1 + t) + (2 + 4t) \, dt = \int_0^1 -1 + 9t \, dt = \int_0^1 2 - t + \frac{9t^2}{2} \bigg|_0^1 = \frac{7}{2}. \]

(8) Sketch the gradient vector field of \( f(x, y) = -xy. \)

\textit{Solution.} Note that \( \nabla f = \langle f_x, f_y \rangle = \langle -y, -x \rangle. \) The vector field looks like the following.

(9) Sketch the gradient vector field of \( f(x, y, z) = -\frac{x^2}{2}. \)

\textit{Solution.} Note that \( \nabla f = \langle f_x, f_y, f_z \rangle = \langle -x, 0, 0 \rangle. \)

(10) Consider a thin plate that occupies the region \( D \) bounded by the parabola \( y = 1 - x^2, \ x = 1 \) and \( y = 1 \) in the first quadrant with density function \( \rho(x, y) = y. \)

(a) Find the mass of the thin plate.

(b) Find the center of mass of the thin plate.

\textit{Solution.} (a) The region of integration is \( R = \{(x, y) | 0 \leq x \leq 1, 1 - x^2 \leq y \leq 1 \}. \)

The mass is \( m = \int \int_R \rho(x, y) \, dA = \int_0^1 \int_{1-x^2}^1 y \, dy \, dx = \int_0^1 \int_{1-x^2}^1 \frac{y^2}{2} \bigg|_{1-x^2}^1 \, dx = \int_0^1 \left( \frac{1}{2} - \frac{(1-x^2)^2}{2} \right) \, dx = \int_0^1 \left( \frac{1}{2} - \frac{x^4 - 2x^2 + 1}{2} \right) \, dx = \int_0^1 \left( \frac{x^4}{2} + x^2 \right) \, dx = \left( \frac{x^5}{10} + \frac{x^3}{3} \right) \bigg|_0^1 = -\frac{1}{10} + \frac{1}{3} = \frac{7}{30}. \)

(b) The center of mass is \( \left( \frac{\int \int_R \rho(x, y) x \, dA}{m}, \frac{\int \int_R \rho(x, y) y \, dA}{m} \right). \)
Now \( \int \int R \rho(x, y) \, x \, dA = \int_0^1 \int_{1-x^2}^1 xy \, dy \, dx + \int_0^1 \int_{1-x^2}^1 x \, y^2 \, dy \, dx = \int_0^1 \frac{x}{2} - x \left( \frac{1-x^2}{2} \right)^2 \, dx = \int_0^1 \frac{x}{2} - x \left( \frac{1-x^2}{2} \right)^2 \, dx = \int_0^1 \frac{x}{2} - \left( \frac{x^5 - 2x^3 + x}{2} \right) \, dx = \int_0^1 \frac{x^5 + 2x^3}{2} \, dx = \left( -\frac{x^6}{12} + \frac{2x^4}{8} \right) \bigg|_0^1 = -\frac{1}{12} + \frac{2}{8} = \frac{1}{6} \).

\( \int \int R \rho(x, y) \, y \, dA = \int_0^1 \int_{1-x^2}^1 y \, dy \, dx + \int_0^1 \int_{1-x^2}^1 y^2 \, dy \, dx = \int_0^1 \frac{y^2}{3} \bigg|_{1-x^2}^1 \, dx = \int_0^1 \frac{1}{3} - \left( \frac{-x^6 + 3x^4 - 3x^2 + 1}{3} \right) \, dx = \left( \frac{x^7}{21} - \frac{x^5}{5} + \frac{x^3}{3} \right) \bigg|_0^1 = \left( \frac{1}{21} - \frac{1}{5} + \frac{1}{3} - \frac{1}{3} \right) = \frac{10}{105} = \frac{2}{21} \).

So the center of mass is \( \left( \frac{5}{7}, \frac{38}{49} \right) \).