Solution to Review Problems for Midterm III

MATH 2850 – 004

The detail of the information about the second midterm can be found at
http://www.math.utoledo.edu/∼mtsui/calc06sp/exam/midterm3.html
You should also review the homework and quiz problems to prepare for the midterm.
This midterm will cover 15.8 Lagrange multipliers , Chapter 16 (except 16.9) and 17.1-17.2.

(1) Use Lagrange multipliers to find the maximum or minimum values of f subject to the given constraint.

(a) $f(x, y) = xy$, $(1 + x^2)(1 + y^2) = 4$.

Solution. The constraint is $g(x, y) = (1+x^2)(1+y^2) = 4$ and the function is $f(x, y) = xy$. We have $\nabla f(x, y) = (y, x)$ and $\nabla g(x, y) = (2x(1+y^2), 2y(1+y^2))$. The equation $\nabla f(x, y) = \lambda \nabla g(x, y)$ is the same as $y = 2\lambda x(1 + x^2)$ and $x = 2\lambda y(1 + y^2)$. Thus the optimizer (x, y) satisfy

$$y = 2\lambda x(1 + x^2) \tag{1}$$

$$x = 2\lambda y(1 + y^2) \tag{2}$$

$$(1 + x^2)(1 + y^2) = 4 \tag{3}$$

Multiplying the first two equations, we have $xy = 4\lambda^2 xy(1 + x^2)(1 + y^2)$. The constraint equation $(1 + x^2)(1 + y^2) = 4$ implies $xy = 16\lambda^2 xy$. Thus $xy = 0$ or $\lambda = \pm\frac{1}{4}$. But $xy = 0$, $y = 2\lambda x(1 + x^2)$ and $x = 2\lambda y(1 + y^2)$ imply $(x, y) = (0, 0)$ which doesn't satisfy the constraint equation. Suppose $\lambda = \frac{1}{4}$, we have $y = \frac{1}{2}x(1 + x^2)$ and $x = \frac{1}{2}y(1 + x^2) = \frac{1}{4}x(1 + x^2)^2$. Thus $x = 0$ or $(1 + x^2)^2 = 4$. Thus $(x, y) = (1, 1), (-1, -1)$. Suppose $\lambda = -\frac{1}{4}$, we have $y = -\frac{1}{2}x(1 + x^2)$ and $x = -\frac{1}{2}y(1 + x^2) = \frac{1}{4}x(1 + x^2)^2$. Thus $x = 0$ or $(1 + x^2)^2 = 4$. This implies $(x, y) = (1, -1), (-1, 1)$. We have $f(1, 1) = f(-1, -1) = 1$ and $f(1, -1) = f(-1, 1) = -1$. Thus the maximizers are $(1, 1)$ and $(-1, -1)$ with maximum 1. The minimizers are $(-1, 1)$ and $(1, -1)$ with minimum -1. □

(b) $f(x, y, z) = x^2 - y^2$, $x^2 + y^2 = 2$

Solution. Let $f(x, y) = x^2 - y^2$ and $g_(x, y) = x^2 + y^2 = 2$. The necessary conditions for the optimizer (x, y) are
$\nabla f(x, y) = \lambda \nabla g_(x, y)$ and the constraint equations $x^2 + y^2 = 2$ which are:

Since $\nabla f(x,y) = (2x, -2y)$ and $\nabla g(x,y) = (2x, 2y)$, thus (x,y) must satisfy

$$2x = 2\lambda x \tag{4}$$
$$-2y = 2\lambda y \tag{5}$$
$$x^2 + y^2 = 2 \tag{6}$$

From (4), (5) , we get $4x^2 + 4y^2 = 4\lambda^2(x^2 + y^2)$. Since $x^2 + y^2 = 2$m we have $\lambda^2 = 1$. So $\lambda = \pm 1$. If $lambda = 1$, then eq(4) is always true and we get $y = 0$ by eq(5). Using $x^2 + y^2 = 2$, we get $x = \pm\sqrt{2}$.
If $lambda = -1$, then eq(5) is always true and we get $x = 0$ by eq(4). Using $x^2 + y^2 = 2$, we get $y = \pm\sqrt{2}$.
So the candidates are $(\sqrt{2}, 0)$, $(-\sqrt{2}, 0)$, $(0, \sqrt{2}, 0)$ and $(0, \sqrt{2}, 0)$.
So $f((\sqrt{2}, 0)) = f((-\sqrt{2}, 0)) = 2$ and $f((0, \sqrt{2}, 0)) = f((0, \sqrt{2}, 0)) = -2$.
Thus the maximum is 2, the minimum is -2, the maximizers are $(\sqrt{2}, 0)$, $(-\sqrt{2}, 0)$, and the minimizers are $(0, \sqrt{2}, 0)$ and $(0, \sqrt{2}, 0)$.

□

(c) $f(x,y,z) = x + y + z$, $x^2 + y^2 + z^2 = 1$.

Solution. Let $f(x,y,z) = x + y + z$ and $g(x,y,z) = x^2 + y^2 + z^2 = 1$.
We have $\nabla f(x,y,z) = (1,1,1)$ and$\nabla g(x,y,z) = (2x, 2y, 2z)$.
The necessary conditions for the optimizer (x,y,z) are
$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ and the constraint equations which are:

$$1 = 2\lambda x \tag{7}$$
$$1 = 2\lambda y \tag{8}$$
$$1 = 2\lambda z \tag{9}$$
$$x^2 + y^2 + z^2 = 1 \tag{10}$$

From (7),(8) (9) and (10), we know that $\lambda \neq 0$, $x = \frac{1}{2\lambda}$, $y = \frac{1}{2\lambda}$ and $z = \frac{1}{2\lambda}$.
Plugging into (10), we get $\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 1$, $\frac{3}{4\lambda^2} = 1$ and $\lambda = \pm\frac{\sqrt{3}}{2}$. So $(x,y,z) = (\frac{1}{2\lambda}, \frac{1}{2\lambda}, \frac{1}{2\lambda}) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ or $(x,y,z) = ((-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.
We have $f((\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})) = \frac{3}{\sqrt{3}} = \sqrt{3}$ and $f((-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})) = -\frac{3}{\sqrt{3}} = -\sqrt{3}$.
Thus the maximizers are $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ with maximum $\sqrt{3}$. The minimizers are $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ with minimum $-\sqrt{3}$. □

(2) Using Riemann sums with two subdivisions in each direction, find upper and lower bounds for the volume under the graph of $f(x, y) = 1 + x^2 + y^2$ above the rectangle R with $0 \le x \le 2$, $0 \le y \le 2$.

Solution. Let $f(x, y) = 1 + x^2 + y^2$. We have $f_x = 2x \ge 0$ and $f_y = 2y \ge 0$ in the rectangle R. The lower estimate is $f(0, 0) \cdot 1 + f(1, 0) \cdot 1 + f(0, 1) \cdot 1 + f(1, 1) \cdot 1 = 1 + 2 + 2 + 3 = 8$.

The upper estimate is $f(1, 1) \cdot 1 + f(1, 2) \cdot 1 + f(2, 1) \cdot 1 + f(2, 2) \cdot 1 = 3 + 5 + 5 + 9 = 22$. □

(3) Compute the following iterated integrals.

(a) $\int\int_D \frac{6x}{y^3+1} dA$ where $D = \{(x, y)|0 \le x \le y, 0 \le y \le 1\}$.

Solution. $\int\int_D \frac{6x}{y^3+1} dA = \int_0^1 \int_0^y \frac{6x}{y^3+1} dxdy = \int_0^1 \frac{3x^2}{y^3+1}\Big|_0^y dx = \int_0^1 6\frac{3y^2}{y^3+1} dy$. Let $u = y^3 + 14$. Then $du = 3y^2 dy$, $y^2 dy = \frac{u}{3} du$ and $\int \frac{3y^2}{y^3+1} dx = \int \frac{3}{3u} du = \ln|u| + C = \ln|y^3 + 1| + C$. Hence $\int\int_D \frac{6x}{y^3+1} dA = \ln|y^3 + 1|\Big|\Big|_0^1 = \ln(2)$.

□

(b) $\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3} dxdy$

Let $D = \{(x, y)|\sqrt{y} \le x \le 1, 0 \le y \le 1\}$ Then $0 \le y \le x^2$ and $0 \le x \le 1$. So D is the same as $\{(x, y)|0 \le x \le 1, 0 \le y \le x^2\}$.

We have $\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3} dxdy = \int_0^1 \int_0^{x^2} \frac{ye^{x^2}}{x^3} dydx = \int_0^1 \frac{y^2 e^{x^2}}{2x^3}\Big|_0^{x^2} dx = \int_0^1 \frac{xe^{x^2}}{2} dx = \frac{e^{x^2}}{4}|_0^1 = \frac{e}{4} - \frac{1}{4}$.

(c) $\int_0^1 \int_x^1 \cos(y^2) dydx$

Solution. Let $D = \{(x, y)|0 \le x \le 1, x \le y \le 1\}$. Since $x \le y$ and $0 \le x$, we have $0 \le x \le y$. Since $x \le y \le 1$ and $0 \le x$, we have $0 \le y \le 1$. So D is the same as $\{(x, y)|0 \le x \le y, 0 \le y \le 1\}$.

We have $\int_0^1 \int_x^1 \cos(y^2) dydx = \int_0^1 \int_0^y \cos(y^2) dxdy = \int_0^1 x\cos(y^2)|_0^x dx = \int_0^1 y\cos(y^2) dy = \frac{\sin(y^2)}{2}|_0^1 = \frac{\sin(1)}{2}$.

□

(d) $\int_{-3}^0 \int_0^{\sqrt{9-y^2}} \sqrt{x^2 + y^2} dxdy$

Solution. The region of integration is $\{(x, y)0 \le x \le \sqrt{9 - y^2}, -3 \le y \le 0\}$. The is the region in fourth quadrant. In polar coordinates, it is $R = \left\{(r, \theta) : 0 \le r \le 3, \frac{-\pi}{2} \le \theta \le 0\right\}$. We also have $(\sqrt{x^2 + y^2} = (r^2)^{\frac{1}{2}} = r$ and

$\int_{-3}^{0}\int_{0}^{\sqrt{9-y^2}}\sqrt{x^2+y^2}dxdy = \int_{\frac{-\pi}{2}}^{0}\int_{0}^{3} r\cdot rdrd\theta$
$= \int_{\frac{-\pi}{2}}^{0}\int_{0}^{3} r^2drd\theta = \int_{0}^{2\pi}\frac{r^2}{3}|_0^3 d\theta = 3\theta|_{\frac{-\pi}{2}}^{0} = \frac{3\pi}{2}$.

□

(e) $\int_0^2\int_{-\sqrt{4-x^2}}^{0} e^{-x^2-y^2}dydx$

Solution. The region of integration is $\{(x,y)0 \le x \le 2, -\sqrt{4-x^2} \le y \le 0\}$. The is the region in fourth quadrant. In polar coordinates, it is $R = \{(r,\theta) : 0 \le r \le 3, \frac{-\pi}{2} \le \theta \le 0\}$. We also have $x^2+y^2 = r^2$ and
$\int_0^2\int_{-\sqrt{4-x^2}}^{0} e^{-x^2-y^2}dydx = \int_{\frac{-\pi}{2}}^{0}\int_0^2 e^{-r^2}\cdot rdrd\theta$
$= \int_{\frac{-\pi}{2}}^{0} -\frac{e^{-r^2}}{2}\Big|_0^2 d\theta = -(\frac{e^{-4}}{2} - \frac{1}{2})\cdot\frac{\pi}{2}$.

□

(f) $\int_0^1\int_0^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyzdzdxdy$

Solution. The region of integration is $\{(x,y,z)0 \le x \le \sqrt{1-y^2}, 0 \le y \le 1, x^2+y^2 \le z \le \sqrt{x^2+y^2}\}$. In cylindrical coordinates, it is $R = \{(r,\theta,z) : 0 \le r \le 1, 0 \le \theta \le \frac{\pi}{2}, r^2 \le z \le r\}$. Recall that $x = r\cos(\theta)$, $x = r\sin(\theta)$ We have $xyz = r\cos(\theta)\cdot r\sin(\theta) = r^2\cos(\theta)\cdot\sin(\theta)$ and
$\int_0^1\int_0^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyzdzdxdy = \int_0^{\frac{\pi}{2}}\int_0^1\int_{r^2}^{r} r^2\cos(\theta)\cdot\sin(\theta)z\cdot rdzdrd\theta$
$= \int_0^{\frac{\pi}{2}}\int_0^1\int_{r^2}^{r} r^3z\cos(\theta)\cdot\sin(\theta)dzdrd\theta = \int_0^{\frac{\pi}{2}}\int_0^1 r^3\frac{z^2}{2}\cos(\theta)\cdot\sin(\theta)\Big|_{r^2}^{r} drd\theta$
$= \int_0^{\frac{\pi}{2}}\int_0^1 r^3\frac{r^2-r^4}{2}\cos(\theta)\cdot\sin(\theta)drd\theta$
$= \int_0^{\frac{\pi}{2}}\int_0^1 (\frac{r^5}{2} - \frac{r^7}{2})\cos(\theta)\cdot\sin(\theta)drd\theta = \int_0^{\frac{\pi}{2}}(\frac{r^6}{12} - \frac{r^8}{16})\cos(\theta)\cdot\sin(\theta)\Big|_0^1 d\theta$
$= \int_0^{\frac{\pi}{2}}\frac{1}{48}\cos(\theta)\cdot\sin(\theta)d\theta$
$= \frac{1}{96}\sin^2(\theta)\Big|_0^{\frac{\pi}{2}} = \frac{1}{96}$.

□

(g) $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{2-x^2-y^2}(x^2+y^2)^{\frac{3}{2}}dzdydx$

Solution. The region of integration is $\{(x,y,z)| -1 \le x \le 1, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, x^2+y^2 \le z \le 2-x^2-y^2\}$. In cylindrical coordinates, it is $R = \{(r,\theta,z) : 0 \le r \le 1, 0 \le \theta \le 2\pi, r^2 \le z \le 2-r^2\}$. Recall that $x = r\cos(\theta)$, $x = r\sin(\theta)$ We have $(x^2+y^2)^{\frac{3}{2}} = r^3$ and
$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{2-x^2-y^2}(x^2+y^2)^{\frac{3}{2}}dzdydx = \int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} r^3\cdot rdzdrd\theta$
$= \int_0^{2\pi}\int_0^1 r^4z\Big|_{r^2}^{2-r^2} drd\theta$
$= \int_0^{2\pi}\int_0^1 r^4(2-2r^2)drd\theta$

$= \int_0^{2\pi} \int_0^1 (\frac{2r^5}{5} - \frac{2r^7}{7})\Big|_0^1 d\theta$
$= \int_0^{\frac{\pi}{2}} \frac{4}{35} d\theta$
$= \frac{8\pi}{35}.$

□

(h) $\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}dzdxdy$

Solution. In spherical coordinates, the region $E = \{(x,y,z)|0 \leq x \leq \sqrt{4-y^2}, -2 \leq y \leq 2, -\sqrt{4-x^2-y^2} \leq z \leq \sqrt{4-x^2-y^2}\}$
is described by the inequalities $0 \leq \rho \leq 2$, $0 \leq \theta \leq \pi\pi$ and $0 \leq \phi \leq \pi$. Note that $y = \rho\sin(\phi)\cos(\theta)$ Hence, the integral is

$$\begin{aligned}
&\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}dzdxdy \\
&= \int_0^{\pi}\int_0^{\pi}\int_0^{2} \rho^2\sin^2(\phi)\cos^2(\theta)(\rho)\ \rho^2\sin(\phi)\ d\rho\ d\theta\ d\phi \\
&= \int_0^{\pi}\int_0^{\pi}\int_0^{2} \rho^5\sin^3(\phi)\cos^2(\theta)\ d\rho\ d\theta\ d\phi \\
&= \left(\int_0^{\pi}\cos^2(\theta)d\theta\right)\left(\int_0^{\pi}\sin^3(\phi)\ d\phi\right)\left(\int_0^{2}\rho^5\ d\rho\right) \\
&= \left(\int_0^{\pi}\frac{1+\cos(2\theta)}{2}d\theta\right)\left(\int_0^{\pi}(1-\cos^2(\phi))\sin(\phi)\ d\phi\right)\left(\int_0^{2}\rho^5\ d\rho\right) \\
&= \left((\frac{\theta}{2}+\frac{\sin(2\theta)}{4})\Big|_0^{\pi}\right)\left((-\cos(\phi)+\frac{\cos^3(\phi)}{3})\Big|_0^{\pi}\right)\left(\frac{\rho^6}{6}\Big|_0^{2}\right) \\
&= \frac{\pi}{2}\cdot\frac{4}{3}\cdot\frac{64}{6} = \frac{64\pi}{9}
\end{aligned}$$

□

(i) $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)dzdxdy$

Solution. In spherical coordinates, the region $E = \{(x,y,z)|0 \leq x \leq \sqrt{9-y^2}, 0 \leq y \leq 3, \sqrt{x^2+y^2} \leq z \leq \sqrt{18-x^2-y^2}\}$
is described by the inequalities $0 \leq \rho \leq \sqrt{18}$, $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi \leq \frac{\pi}{4}$. Note that $\sqrt{x^2+y^2} = \sqrt{18-x^2-y^2}$ if $x^2+y^2 = 9$ and $z = \sqrt{x^2+y^2}$ is $\phi = \frac{\pi}{4}$ in spherical coordinates.

Note that $x^2 + y^2 + z^2 = \rho^2$ Hence, the integral is

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2 + y^2 + z^2) dz dx dy$$
$$= \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{18}} \rho^2 \cdot \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$$
$$= \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{18}} \rho^4 \sin(\phi)\, d\rho\, d\theta\, d\phi$$
$$= \left(\int_0^{\frac{\pi}{2}} d\theta\right) \left(\int_0^{\frac{\pi}{4}} \sin(\phi)\, d\phi\right) \left(\int_0^{\sqrt{18}} \rho^4\, d\rho\right)$$
$$= \left(\frac{\pi}{2}\right) \left(-\cos(\phi))\Big|_0^{\frac{\pi}{4}}\right) \left(\frac{\rho^5}{5}\Big|_0^{\sqrt{18}}\right)$$
$$= \frac{\pi}{2} \cdot (-\frac{1}{\sqrt{2}} + 1) \cdot \frac{(18)^2 \cdot \sqrt{18}}{5}$$

□

(4) Find the volume of the following regions:

(a) The solid bounded by the surface $z = x\sqrt{x^2 + y}$ and the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$ and $z = 0$.

Solution. The volume is $\int_0^1 \int_0^1 x\sqrt{x^2 + y} dx dy$ Let $u = x^2 + y$. Then $du = 2x dx$, $x dx = \frac{du}{2}$ and $\int x\sqrt{x^2 + y} dx = \int \frac{u^{1/2}}{2} du = \frac{u^{3/2}}{3} + C = \frac{(x^2+y)^{3/2}}{3} + C$.
So $\int_0^1 \int_0^1 x\sqrt{x^2 + y} dx dy = \int_0^1 \frac{(x^2+y)^{3/2}}{3}\Big|_0^1 dy$
$= \int_0^1 \frac{(1+y)^{3/2}}{3} - \frac{(y)^{3/2}}{3} dx = \frac{2(1+y)^{5/2}}{15} - \frac{2(y)^{5/2}}{15}\Big|_0^1 = \frac{2(2)^{5/2}}{15} - \frac{2}{15} - (\frac{2}{15} - 0)$
$= \frac{2(2)^{5/2}}{15} - \frac{4}{15} = \frac{8\sqrt{2}}{15} - \frac{4}{15}$ □

(b) The solid that lies between the sphere $x^2 + y^2 + z^2 = 4$, above the $x - y$ plane, and below the cone $z = \sqrt{x^2 + y^2}$.

Solution. %begincenter
The region is bounded above by the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and below by the cone $z = \sqrt{x^2 + y^2}$. We have $\sqrt{x^2 + y^2} \leq z \leq \sqrt{4 - x^2 - y^2}$. Thus $x^2 + y^2 \leq z^2 \leq 4 - x^2 - y^2$ and $x^2 + y^2 \leq 2$
In polar coordinates, this region $x^2+y^2 \leq 2$ is $R = \{(r, \theta) : 0 \leq r \leq \sqrt{2}, 0 \leq \theta \leq 2\pi\}$. Note that $\sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2}$ and $\sqrt{x^2 + y^2} = \sqrt{r^2} = r$.
Hence, we can compute the volume of the region by finding the volume under the graph of $\sqrt{4 - r^2}$ above the disk $R = \{(r, \theta) : 0 \leq r \leq \sqrt{2}, 0 \leq$

$\theta \leq 2\pi\}$ and subtracting the volume under the graph of r above R. Therefore, we have

$$\begin{aligned}
\text{Volume} &= \int_0^{2\pi}\int_0^{\sqrt{2}} \left(\sqrt{4-r^2}\right) r dr\, d\theta - \int_0^{2\pi}\int_0^{2} \left(r\right) r dr\, d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{2}} r\sqrt{4-r^2} - r^2\, dr\, d\theta = \int_0^{2\pi} \left[-\tfrac{1}{3}(4-r^2)^{3/2} - \tfrac{1}{3}r^3\right]_0^{\sqrt{2}}\, d\theta \\
&= \tfrac{1}{3}\int_0^{2\pi} \left(-4\sqrt{2}+8\right)\, d\theta = \tfrac{1}{3}(-4\sqrt{2}+8)\int_0^{2\pi}\, d\theta \\
&= \tfrac{1}{3}(-8\sqrt{2}\pi + 16\pi)\,.
\end{aligned}$$

□

(c) The solid bounded by the plane $x+y+z=3$, $x=0$, $y=0$ and $z=0$.

Solution. The region E bounded by the xy, yz, xz planes and the plane $x+y+z=3$ is the set $\{(x,y,z) \in \mathbb{R}^3 : 0 \leq x \leq 3, 0 \leq y \leq 3-x, 0 \leq z \leq 3-x-y\}$. The volume of E is

$$\begin{aligned}
&\int\int\int_E dV = \int_0^{3)}\int_0^{3-x}\int_0^{3-x-y} dz\, dy\, dx = \int_0^{3)}\int_0^{3-x} z\Big|_0^{3-x-y} dy\, dx \\
&= \int_0^{3}\int_0^{3-x} 3-x-y\, dy\, dx = \int_0^{3} 3y - xy - \frac{y^2}{2}\Big|_0^{3-x} dx \text{ (by substitution u=4-x-2y)} \\
&= \int_0^{3} 3(3-x) - x(3-x) - \frac{(3-x)^2}{2}\, dx = \int_0^{3} 9 - 3x + 3 - 3x + x^2 - \frac{(x^2-6x+9)}{2}\, dx \\
&\qquad = \int_0^{3} \frac{9}{2} - 3x + \frac{x^2}{2}\, dx = \frac{9x}{2} - \frac{3x^2}{2} + \frac{x^3}{6}\Big|_0^{3} = \frac{27}{2} - \frac{27}{2} + \frac{27}{6} = \frac{9}{2}.
\end{aligned}$$

□

(d) The region bounded by the cylinder $x^2+y^2=4$ and the plane $z=0$ and $y+z=3$.

Solution. The region is bounded above by the plane $z=3-y$ and below by $z=0$. In polar coordinates, this region $x^2+y^2 \leq 4$ is $R = \{(r,\theta) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$. Note that $z = 3-y = 3-r\cos(\theta)$ Hence, we can compute the volume of the region by

$$\text{Volume} = \int_0^{2\pi}\int_0^2 \big(3 - r\cos(\theta)\big)\, r dr\, d\theta$$
$$= \int_0^{2\pi}\int_0^2 3r - r^2\cos(\theta)\, dr\, d\theta = \int_0^{2\pi} \left[\tfrac{3}{2}r^2 - \tfrac{1}{3}r^3\cos(\theta)\right]_0^2 d\theta$$
$$= \int_0^{2\pi} \left[6 - \tfrac{8}{3}\cos(\theta)\right] d\theta \ = 12\pi\,.$$

□

(5) Find the area of the following surfaces.

(a) The cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$

Solution. We have $z = f(x, y) = \sqrt{x^2 + y^2}$, $f_x = \frac{x}{\sqrt{x^2+y^2}}$, $f_y = \frac{y}{\sqrt{x^2+y^2}}$ and $\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + (\frac{x}{\sqrt{x^2+y^2}})^2 + (\frac{y}{\sqrt{x^2+y^2}})^2} = \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}} = \sqrt{2}$. Since $1 \le z = \sqrt{x^2 + y^2} \le 2$, we have $1 \le x^2 + y^2 \le 4$. The region $E = \{(x, y)|1 \le x^2 + y^2 \le 4\}$ is $\{(r, \theta)|0 \le r \le 2, 0 \le \theta \le 2\pi\}$ in polar coordinates. Hence the area of surface is
$\int\int_E \sqrt{1 + f_x^2 + f_y^2} dxdy = \int_0^{2\pi}\int_0^2 r dr d\theta = \int_0^{2\pi}\int_0^2 \sqrt{2}\frac{r^2}{2}|_0^2 dr d\theta = 2\sqrt{2}\cdot 2\pi = 4\sqrt{2}\pi$.

□

(b) Given by $\{(x, y, z)|x^2 + y^2 = 1, 0 \le z \le xy, x \ge 0, y \ge 0\}$.

Solution. Note that the curve $x^2+y^2 = 1$, $x \ge 0$ and $y \ge 0$ can be parameterized by $x = \cos(t)$, $y = \sin(t)$ with $0 \le t \le \frac{\pi}{2}$. So $ds = \sqrt{x'(t)^2 + y'(t)^2}dt = dt$ and $z = xy = \cos(t)\sin(t)$. So the area is $\int_0^{\frac{\pi}{2}} xyds = \int_0^{\frac{\pi}{2}} \cos(t)\sin(t)dt = \frac{\sin^2(t)}{2}|_0^{\frac{\pi}{2}} = \frac{1}{2}$. □

(6) Rewrite the integral $\int_{-1}^1\int_{x^2}^1\int_0^{1-y} f(x, y, z)dzdydx$ as an iterated integral in the order of $dxdydz$

Solution. The region of integration is $E = \{(x, y, z)|0 \le z \le 1 - y, x^2 \le y \le 1, -1 \le x \le 1\}$. Since $x^2 \le y$ and $-1 \le x$, we have $-1 \le x \le \sqrt{y}$. Using $z \le 1 - y$, $x^2 \le y$ and $-1 \le x \le 1$, we have $0 \le y \le 1 - z$. Using $0 \le z \le 1 - y$ and $0 \le y$, we have $1 - y \le 1$ and $0 \le z \le 1$.
So $\int_{-1}^1\int_{x^2}^1\int_0^{1-y} f(x, y, z)dzdydx = \int_0^1\int_0^{1-z}\int_{-1}^{\sqrt{y}} f(x, y, z)dxdydz$

□

(7) Evaluate the following line integrals.

(a) $\int_C 2zds$ where C is given by $x = \cos(t)$, $y = \sin(t)$, $z = t$, $0 \leq \pi$.

Solution. Note that $\frac{dx}{dt} = \frac{d}{dt}(\cos(t)) = -\sin(t)$, $\frac{dy}{dt} = \frac{d}{dt}(\sin(t)) = \cos(t)$ and $\frac{dz}{dt} = \frac{d}{dt}(t) = 1$.
Since $x = \cos(t)$, $y = \sin(t)$ and $z = t$,
$\int_C 2zds = \int_0^{2\pi} 2t\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}dt$
$= \int_0^{2\pi} 2t\sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1}dt = \int_0^{2\pi} 2\sqrt{2}tdt = 2\sqrt{2}\frac{t^2}{2}|_0^{2\pi} = 2\sqrt{2} \cdot 2\pi^2 = 4\sqrt{2}\pi^2$. □

(b) Evaluate $\int_C F \cdot dr$ where $F =< y, x >$ and C is given by $r(t) = (t, t^2)$ where $0 \leq t \leq 1$.

Solution. Note that $r(t) = (t, t^2)$ with $0 \leq t \leq 1$ is part of the parabola between $(0, 0)$ and $(1, 1)$. Also $r'(t) = (x'(t), y'(t)) = (1, 2t)$.
$\int_C F \cdot dr = \int_0^1 (y(t), x(t)) \cdot (x'(t), y'(t))dt = \int_0^1 (t^2, t) \cdot (1, 2t)dt = \int_0^1 3t^2dt = t^3|_0^1 = 1$

□

(c) $\int_C ydx + xdy$ where C is the line between $A = (1, 1)$ and $B = (-1, 5)$.

Solution. The equation of the line between $A = (1, 1)$ and $B = (-1, 5)$ is $r(t) = (x(t), y(t)) = (1 - 2t, 1 + 4t)$ with $0 \leq t \leq 1$. Also $r'(t) = (x'(t), y'(t)) = (-2, 4)$.
$\int_C ydx + xdy = \int_0^1 y(t)x'(t) + x(t)y'(t)dt = \int_0^1 (1 + 4t) \cdot (-2) + (1 - 2t) \cdot 4dt = \int_0^1 -2 - 8t + 4 - 8tdt = \int_0^1 2 - 16tdt = 2t - 8t^2|_0^1 = -6$.

□

(d) $\int_C F \cdot dr$ where $F =< x, y, z >$ and C is given by $r(t) = (\cos(t), \sin(t), t)$ where $0 \leq t \leq \pi$.

Solution. Note that $\frac{dx}{dt} = \frac{d}{dt}(\cos(t)) = -\sin(t)$, $\frac{dy}{dt} = \frac{d}{dt}(\sin(t)) = \cos(t)$ and $\frac{dz}{dt} = \frac{d}{dt}(t) = 1$.
Since $x = \cos(t)$, $y = \sin(t)$ and $z = t$,
$\int_C F \cdot dr = \int_0^{\pi} (\cos(t), \sin(t), t) \cdot (-\sin(t), \cos(t), 1)dt = \int_0^{\pi} tdt = \frac{\pi^2}{2}$

□

(e) $\int_C F \cdot dr$ where $F =< x, y, z >$ and C is the line between $A = (1, 1, 1)$ and$B = (-1, 0, 3)$.

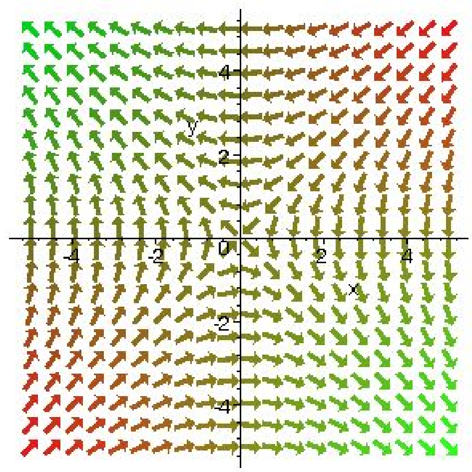
Solution. The equation of the line between $A = (1, 1, 1)$ and$B = (-1, 0, 3)$ is $r(t) = (x(t), y(t), z(t)) = (1 - 2t, 1 - t, 1 + 2t)$ with $0 \leq t \leq 1$. Also $r'(t) = (x'(t), y'(t), z'(t)) = (-2, -1, 2)$.

$\int_C F \cdot dr = \int_0^1 (1-2t, 1-t, 1+2t) \cdot (-2,-1,2)dt = \int_0^1 (-2+4t) + (-1+t) + (2+4t)dt = \int_0^1 -1 + 9tdt = \int_0^1 2 - t + \frac{9t^2}{2}|_0^1 = \frac{7}{2}$.

□

(8) Sketch the gradient vector field of $f(x,y) = -xy$.

Solution. Note that $\nabla f =< f_x, f_y >=< -y, -x >$. The vector field looks like the following.



□

(9) Sketch the gradient vector field of $f(x,y,z) = -\frac{x^2}{2}$.

Solution. Note that $\nabla f =< f_x, f_y, f_z >=< -x, 0, 0 >$. □

(10) Consider a thin plate that occupies the region D bounded by the parabola $y = 1 - x^2$, $x = 1$ and $y = 1$ in the first quadrant with density function $\rho(x,y) = y$.

(a) Find the mass of the thin plate.

(b) Find the center of mass of the thin plate.

Solution. (a) The region of integration is $R = \{(x,y)|0 \le x \le 1, 1 - x^2 \le y \le 1\}$.

The mass is $m = \int\int_R \rho(x,y)dA = \int_0^1 \int_{1-x^2}^1 ydydx = \int_0^1 \int_{1-x^2}^1 \frac{y^2}{2}|_{1-x^2}^1 dx = \int_0^1 (\frac{1}{2} - \frac{(1-x^2)^2}{2})dx = \int_0^1 (\frac{1}{2} - \frac{x^4 - 2x^2 + 1}{2})dx = \int_0^1 (-\frac{x^4}{2} + x^2)dx = (-\frac{x^5}{10} + \frac{x^3}{3})|_0^1 = -\frac{1}{10} + \frac{1}{3} = \frac{7}{30}$.

(b) The center of mass $= (\frac{\int\int_R \rho(x,y)xdA}{m}, \frac{\int\int_R \rho(x,y)ydA}{m}$.

Now $\int\int_R \rho(x,y)x dA = \int_0^1 \int_{1-x^2}^1 xydydx \int_0^1 \int_{1-x^2}^1 x\frac{y^2}{2}|_{1-x^2}^1 dx = \int_0^1 \frac{x}{2} - x\frac{(1-x^2)^2}{2}dx = \int_0^1 \frac{x}{2} - x \cdot \frac{(1-x^2)^2}{2}dx = \int_0^1 \frac{x}{2} - (\frac{x^5-2x^3+x}{2})dx = \int_0^1 \frac{-x^5+2x^3}{2}dx = (-\frac{x^6}{12} + \frac{2x^4}{8})|_0^1 = -\frac{1}{12} + \frac{2}{8} = \frac{1}{6}$.
$\int\int_R \rho(x,y)y dA = \int_0^1 \int_{1-x^2}^1 y \cdot ydydx = \int_0^1 \int_{1-x^2}^1 y^2 dydx = \int_0^1 \frac{y^3}{3}\Big|_{1-x^2}^1 dx = \int_0^1 \frac{1}{3} - \frac{(1-x^2)^3}{3}dx = \int_0^1 \frac{1}{3} - \frac{(-x^6+3\,x^4-3\,x^2+1)}{3}dx == \frac{x^7}{21} - \frac{x^5}{5} + \frac{x^3}{3})|_0^1 = (\frac{1}{21} - \frac{1}{5} + \frac{1}{3} - \frac{1}{3}) = \frac{19}{105}$.
So the center of mass is $(\frac{\frac{1}{6}}{\frac{7}{30}}, \frac{\frac{19}{105}}{\frac{7}{30}}) = (\frac{5}{7}, \frac{38}{49})$.

□