Solution to Problem Set #2

1. Using vectors, prove that the diagonals of a parallelogram are penpendicular if and only if the parallelogram is a rhombus.(Note: A **rhombus** is a parallelogram whose four sides all have the same length.)

Solution. Let $\overrightarrow{a}$ and $\overrightarrow{b}$ be vectors along two sides of the parallelogram. The diagonal vectors of the rhombus can be expressed as $\overrightarrow{a} + \overrightarrow{b}$ and $\overrightarrow{a} - \overrightarrow{b}$. Since $\overrightarrow{a} \cdot \overrightarrow{b} = \overrightarrow{b} \cdot \overrightarrow{a}$, we have

$$(\overrightarrow{a}+\overrightarrow{b})\cdot(\overrightarrow{a}-\overrightarrow{b}) = \overrightarrow{a}\cdot\overrightarrow{a}-\overrightarrow{a}\cdot\overrightarrow{b}+\overrightarrow{b}\cdot\overrightarrow{a}-\overrightarrow{b}\cdot\overrightarrow{b} = \overrightarrow{a}\cdot\overrightarrow{a}-\overrightarrow{b}\cdot\overrightarrow{b} = |\overrightarrow{a}|^2-|\overrightarrow{b}|^2.$$

Thus $(\overrightarrow{a} + \overrightarrow{b}) \cdot (\overrightarrow{a} - \overrightarrow{b}) = 0$ if and only if $|\overrightarrow{a}| = |\overrightarrow{b}|$. Therefore $\vec{a} + \vec{b}$ is perpendicular to $\vec{a} - \vec{b}$ if and only if all fours sides of the parallelogram are the same length, i.e. the diagonals of a parallelogram are penpendicular if and only if the parallelogram is a rhombus □

2. Suppose $\vec{u}$ and $\vec{v}$ are nonzero vectors. Show that $||\vec{v}||\vec{u} + ||\vec{u}||\vec{v}$ bisects the angle between $\vec{u}$ and $\vec{v}$.

Solution. Let $\vec{w} = ||\vec{v}||\vec{u} + ||\vec{u}||\vec{v}$.
Let θ_1 be the angle between the vector $\vec{u}$ and $\vec{w}$.
Let θ_2 be the angle between the vector $\vec{v}$ and $\vec{w}$.
By the property of the dot product, we have
$\cos(\theta_1) = \frac{\vec{u}\cdot\vec{w}}{||\vec{u}||\ ||\vec{w}||}$ and $\cos(\theta_2) = \frac{\vec{v}\cdot\vec{w}}{||\vec{v}||\ ||\vec{w}||}$.
Note that
$\vec{u} \cdot \vec{w} = \vec{u} \cdot (||\vec{v}||\vec{u} + ||\vec{u}||\vec{v}) = ||\vec{v}||\vec{u} \cdot \vec{u} + ||\vec{u}||\vec{u} \cdot \vec{v} = ||\vec{v}||\ ||\vec{u}||^2 + ||\vec{u}||\vec{u} \cdot \vec{v}$
and $\vec{v} \cdot \vec{w} = \vec{v} \cdot (||\vec{v}||\vec{u} + ||\vec{u}||\vec{v}) = ||\vec{v}||\vec{v} \cdot \vec{u} + ||\vec{u}||\vec{v} \cdot \vec{v} = ||\vec{v}||\ \vec{v} \cdot \vec{u} + ||\vec{u}||\ ||\vec{v}||^2$.
Thus $\cos(\theta_1) = \frac{||\vec{v}||\cdot||\vec{u}||^2+||\vec{u}||\vec{u}\cdot\vec{v}}{||\vec{u}||\ ||\vec{w}||} = \frac{||\vec{v}||\ ||\vec{u}||+\vec{u}\cdot\vec{v}}{||\vec{w}||}$ and
$\cos(\theta_2) = \frac{||\vec{v}||\ \vec{v}\cdot\vec{u}+||\vec{u}||\ ||\vec{v}||^2}{||\vec{v}||\cdot||\vec{w}||} = \frac{\vec{v}\cdot\vec{u}+||\vec{u}||\ ||\vec{v}||}{||\vec{w}||}$.
Since $\vec{v} \cdot \vec{u} = \vec{u} \cdot \vec{v}$ and $||\vec{v}||\ ||\vec{u}|| = ||\vec{u}||\ ||\vec{v}||$, we have $\cos(\theta_1) = \cos(\theta_2)$. This implies that the angle between $\vec{u}$ and $\vec{w} = ||\vec{v}||\vec{u} + ||\vec{u}||\vec{v}$ is the same as the angle between $\vec{v}$ and $\vec{w} = ||\vec{v}||\vec{u} + ||\vec{u}||\vec{v}$. Thus $||\vec{v}||\vec{u} + ||\vec{u}||\vec{v}$ bisects the angle between $\vec{u}$ and $\vec{v}$. □

3. Let $\overrightarrow{u} = 2j$ and let $\overrightarrow{v}$ be a vector with length 9 that starts at the origin and rotates in the xy-plane. Find the maximum and minimum values of $\overrightarrow{u} \cdot \overrightarrow{v}$.

Proof. Recall that $\overrightarrow{u} \cdot \overrightarrow{v} = |\overrightarrow{u}||\overrightarrow{v}|\cos(\theta)$. From $\overrightarrow{u} = 2j$,
we get $|\overrightarrow{u}| = |< 0, 2, 0 >| = 2$. Since $\overrightarrow{u}$ and $\overrightarrow{v}$ lies on the same plane, the angle between them is between 0 and π. Thus $-1 \leq \cos(\theta) \leq 1$. Using $|\overrightarrow{u}| = 2$ and $|\overrightarrow{v}| = 9$, we have $-18 \leq \overrightarrow{u} \cdot \overrightarrow{v} = |\overrightarrow{u}||\overrightarrow{v}|\cos(\theta) =$

$18\cos(\theta) \leq 18$. The maximum of $\overrightarrow{u} \cdot \overrightarrow{v}$ is 18 and the minimum of $\overrightarrow{u} \cdot \overrightarrow{v}$ is -18.

□

4. **(a)** Suppose that the area of the parallelogram spanned by the vectors $\vec{u}$ and $\vec{v}$ are 10. What is the area of the parallelogram spanned by the vectors $2\vec{u} + 3\vec{v}$ and $-3\vec{u} + 4\vec{v}$?

(b) Given $(\vec{u} \times \vec{v}) \cdot \vec{w} = 10$. What is $\Big((\vec{u} + \vec{v}) \times (\vec{v} + \vec{w})\Big) \cdot (\vec{w} + \vec{u})$?

Solution. The area of the parallelogram spanned by the vectors $\vec{u}$ and $\vec{v}$ is $||\vec{u} \times \vec{v}| = 10$. We know that the area of the parallelogram spanned by the vectors $2\vec{u}+3\vec{v}$ and $-3\vec{u}+4\vec{v}$ is $||(2\vec{u}+3\vec{v}) \times (-3\vec{u}+4\vec{v})||$. Note that $(2\vec{u} + 3\vec{v}) \times (-3\vec{u} + 4\vec{v}) = 2\vec{u} \times (-3\vec{u} + 4\vec{v}) + 3\vec{v} \times (-3\vec{u} + 4\vec{v})$
$= -6\vec{u} \times \vec{u} + 8\vec{u} \times \vec{v} - 9\vec{v} \times \vec{u} + 12\vec{v} \times \vec{v} = 8\vec{u} \times \vec{v} + 9\vec{u} \times \vec{v} = 17\vec{u} \times \vec{v}$.
We have used the fact that $\vec{u} \times \vec{u} = \vec{v} \times \vec{v} = \vec{0}$ and $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$.
Hence $||(2\vec{u}+3\vec{v}) \times (-3\vec{u}+4\vec{v})|| = ||17\vec{u}\times\vec{v}|| = 17||\vec{u}\times\vec{v}| = 170$. Therefore the area of the parallelogram spanned by the vectors $2\vec{u} + 3\vec{v}$ and $-3\vec{u} + 4\vec{v}$ is 170.

By distributive law, we have
$\Big((\vec{u}+\vec{v}) \times (\vec{v}+\vec{w})\Big) \cdot (\vec{w}+\vec{u}) = \Big((\vec{u}\times (\vec{v}+\vec{w})\Big) \cdot (\vec{w}+\vec{u}) + \Big((\vec{v}\times (\vec{v}+\vec{w})\Big) \cdot (\vec{w}+\vec{u}) =$
$\Big(\vec{u} \times \vec{v}\Big) \cdot (\vec{w} + \vec{u}) + \Big(\vec{u} \times \vec{w}\Big) \cdot (\vec{w} + \vec{u}) + \Big((\vec{v} \times \vec{v}\Big) \cdot (\vec{w} + \vec{u}) + \Big((\vec{v} \times \vec{w}\Big) \cdot (\vec{w} + \vec{u}) =$
$\Big(\vec{u} \times \vec{v}\Big) \cdot \vec{w} + \Big(\vec{u} \times \vec{v}\Big) \cdot \vec{u} + \Big(\vec{u} \times \vec{w}\Big) \cdot \vec{w} + \Big(\vec{u} \times \vec{w}\Big) \cdot \vec{u} + \Big((\vec{v} \times \vec{w}\Big) \cdot \vec{w} + \Big((\vec{v} \times \vec{w}\Big) \cdot \vec{u}$
where we have used the fact that $\vec{v} \times \vec{v} = \vec{0}$. Furthermore, we have $\Big(\vec{u} \times \vec{v}\Big) \cdot \vec{u} = \Big(\vec{u} \times \vec{w}\Big) \cdot \vec{w} = \Big(\vec{u} \times \vec{w}\Big) \cdot \vec{u} = 0$ and $\Big((\vec{v} \times \vec{w}\Big) \cdot \vec{u} = \Big(\vec{u} \times \vec{v}\Big) \cdot \vec{w}$.

The expression $\Big(\vec{u} \times \vec{v}\Big) \cdot \vec{w} + \Big(\vec{u} \times \vec{v}\Big) \cdot \vec{u} + \Big(\vec{u} \times \vec{w}\Big) \cdot \vec{w} + \Big(\vec{u} \times \vec{w}\Big) \cdot \vec{u} + \Big((\vec{v} \times \vec{w}\Big) \cdot \vec{w} + \Big((\vec{v} \times \vec{w}\Big) \cdot \vec{u}$ can be simplified as $2\Big(\vec{u} \times \vec{v}\Big) \cdot \vec{w}$.
Therefore $\Big((\vec{u} + \vec{v}) \times (\vec{v} + \vec{w})\Big) \cdot (\vec{w} + \vec{u}) = 2\Big(\vec{u} \times \vec{v}\Big) \cdot \vec{w} = 2 \cdot 10 = 20$.

□