Solution to Problem Set #5

1. (20 pts) Show that $v(x,y,t) = \frac{1}{t}e^{(\frac{-x^2-y^2}{4t})}$ is a solution of the heat equation
$v_t = v_{xx} + v_{yy}$. (Hint: Find v_t, v_{xx} and v_{yy} first. Then verify $v_t - v_{xx} - v_{yy} = 0$.)

Solution. We have

$$v_t = (-\frac{1}{t^2} + \frac{x^2+y^2}{4t^3}) \cdot e^{(\frac{-x^2-y^2}{4t})},$$

$$v_x = -\frac{x}{2t^2} \cdot e^{(\frac{-x^2-y^2}{4t})}, \quad v_{xx} = (-\frac{1}{2t^2} + \frac{x^2}{4t^3}) \cdot e^{(\frac{-x^2-y^2}{4t})},$$

$$v_y = -\frac{y}{2t^2} \cdot e^{(\frac{-x^2-y^2}{4t})}, \quad v_{yy} = (-\frac{1}{2t^2} + \frac{y^2}{4t^3}) \cdot e^{(\frac{-x^2-y^2}{4t})}$$

and

$$v_{xx} + v_{yy} = (-\frac{1}{t^2} + \frac{x^2+y^2}{4t^3}) \cdot e^{(\frac{-x^2-y^2}{4t})}.$$

Note that $v(x,y,t) = v(y,x,t)$. The formula of v_{yy} can be obtained from v_{xx} by exchanging x and y. Thus we have $v_t - v_{xx} - v_{yy} = 0$. □

2. **(a)** (10 pts) Find an equation of the tangent plane to the surface $z = f(x,y) = \sqrt{x^2+y^2}$ at the point $(5,12,13)$

(b) (10 pts) Find the linear approximation of the function $f(x,y) = \sqrt{x^2+y^2}$ at $(5,12)$ and use it to estimate $\sqrt{(5.1)^2+(11.9)^2}$.

Solution. The partial derivatives are $f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}}$, $f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$, $f_x(5,12) = \frac{5}{13}$ and $f_x(5,12) = \frac{12}{13}$.
The linear approximation of $f(x,y)$ at $(5,12)$ is

$$\begin{aligned} L(x,y) &= f(5,12) + f_x(5,12)(x-5) + f_y(5,12)(y-12) \\ &= 13 + \frac{5}{13}(x-5) + \frac{12}{13}(y-12). \end{aligned}$$

Thus $L(5.1, 11.9) = 13 + \frac{5}{13} \cdot 0.1 + \frac{12}{13}(-0.1) = 13 - \frac{0.7}{13} \approx 12.946$. Hence $\sqrt{(5.1)^2+(11.9)^2}$ is about 12.946.

□

3. (16 pts) Let $z = f(x,y)$, $x = r\cos(\theta)$ and $y = r\sin(\theta)$.
Show that

$$(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 = (\frac{\partial z}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial z}{\partial \theta})^2.$$

Solution. Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$, the Chain rule yields:

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \cos(\theta)\frac{\partial z}{\partial x} + \sin(\theta)\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin(\theta)\frac{\partial z}{\partial x} + r\cos(\theta)\frac{\partial z}{\partial y}$$

Squaring these equation gives:

$$\begin{aligned}(\frac{\partial z}{\partial r})^2 &= [\cos(\theta)\frac{\partial z}{\partial x} + \sin(\theta)\frac{\partial z}{\partial y}]^2 \\ &= \cos^2(\theta)(\frac{\partial z}{\partial x})^2 + 2\sin(\theta)\cos(\theta)\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + \sin^2(\theta)(\frac{\partial z}{\partial y})^2\end{aligned}$$

$$\begin{aligned}(\frac{\partial z}{\partial \theta})^2 &= (r\sin(\theta)\frac{\partial z}{\partial x} + r\cos(\theta)\frac{\partial z}{\partial y})^2 \\ &= r^2\sin^2(\theta)(\frac{\partial z}{\partial x})^2 - 2r^2\sin(\theta)\cos(\theta)\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + r^2\cos^2(\theta)(\frac{\partial z}{\partial y})^2.\end{aligned}$$

Therefore, we have

$$(\frac{\partial z}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial z}{\partial \theta})^2 = (\sin^2(\theta) + \cos^2(\theta))\Big[(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2\Big] = (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2.$$

□

4. **(a)** (10 pts) Find equations of the tangent plane and the normal line to the surface $2x^2 = y^2 + z^2 - 12$ at the point $(2, -4, 2)$.

(b) (10 pts) Find the points on the surface $2x^2 = y^2 + z^2 - 12$ where the tangent plane is parallel to the plane $4x + 4y + 2z = 1$.

Solution. (a) In general, the normal vector for the tangent plane to the level surface of $F(x, y, z) = k$ at the point (a, b, c) is $\nabla F(a, b, c)$.

The surface $2x^2 = y^2 + z^2 - 12$ can be rewritten as $2x^2 - y^2 - z^2 = -12$. We have $F(x, y, z) = 2x^2 - y^2 - z^2 = -12$, $\nabla F(x, y, z) = \langle 4x, -2y, -2z\rangle$ and $\nabla F(2, -4, 2) = \langle 8, 8, -4\rangle$ Thus the equation of the tangent plane to the surface $2x^2 = y^2 + z^2 - 12$ at the point $(2, -4, 2)$ is
$\langle 8, 8, -4\rangle \cdot \langle x - 2, y + 4, z - 2\rangle = 0$ which yields
$8x - 16 + 8y + 32 - 4z + 8 = 0$. It can be simplified as $8x + 8y - 4z + 24 = 0$ or $2x + 2y - z + 6 = 0$.
The normal line equation at $(2, -4, 2)$ is $x = 2 + 8t$, $y = -4 + 8t$ and $z = 2 - 4t$.

(b) The normal vector for the tangent plane to $F(x, y, z) = 2x^2 - y^2 - z^2 = -12$ at the point (a, b, c) is $\nabla F(a, b, c) = \langle 4a, -2b, -2c\rangle$. The points (a, b, c) on the surface above where the tangent plane is parallel to

$4x+4y+2z=1$ should have its normal vector parallel to $\langle 4,4,2\rangle$. Thus $\langle 4a,-2b,-2c\rangle$ is parallel to $\langle 4,4,2\rangle$. We $\frac{4a}{4}=\frac{-2b}{4}=\frac{-2c}{2}$. Hence $b=-2a$, $c=-a$. Since $(a,b,c)=(a,-2a,-a)$ lies on the surface

$$2x^2-y^2-z^2=-12$$

, we get $2a^2-4a^2-a^2=-12$, i.e $3a^2=12$ and $a=\pm 2$. The points we are looking for are $(a,b,c)=(2,-4,-2)$ or $(a,b,c)=(-2,4,2)$.

□

5. (24 pts) The temperature at any point in the plane is given by the function

$$T(x,y)=\frac{10}{x^2+y^2+1}\,.$$

(a) Find $\nabla T(x,y)$.
(b) Find the direction of the greatest increase in temperature at the point $(3,4)$. What is the magnitude of the greatest increase?
(c) Find the direction of the greatest decrease in temperature at the point $(3,4)$. What is the magnitude of the greatest decrease?
(d) Find a direction at the point $(3,4)$ in which the temperature does not increase or decrease.
(e) Find the rate of change of T at $(3,4)$ in the direction $\langle 5,12\rangle$.
(f) Find the directional derivative of T at $(3,4)$ in the direction $\langle 4,3\rangle$.

Solution. (a) We have

$$\nabla T(x,y)=\langle T_x(x,y),T_y(x,y)\rangle=\langle -\frac{20x}{(x^2+y^2+1)^2},-\frac{20y}{(x^2+y^2+1)^2}\rangle.$$

(b) Since

$$\nabla T(x,y)=\left(-\frac{20x}{(x^2+y^2+1)^2}\right)\vec{i}+\left(-\frac{20y}{(x^2+y^2+1)^2}\right)\vec{j}\,,$$

the direction of the greatest increase in temperature is

$$\nabla T(3,4)=-\frac{60}{676}\vec{i}-\frac{80}{676}\vec{j}=-\frac{15}{169}\vec{i}-\frac{20}{169}\vec{j}=-\frac{5}{169}(3\vec{i}+4\vec{j})$$

and the magnitude is $\|\nabla T(3,4)\|=\frac{5}{169}\sqrt{(3)^2+(4)^2}=\frac{25}{169}$.

(c) The direction of greatest decrease in temperature is

$$-\nabla T(3,4)=\frac{5}{169}(3\vec{i}+4\vec{j})$$

which is a positive scalar multiply of $3\vec{i}+4\vec{j}$.
The magnitude is $\|-\nabla T(3,4)\| = \frac{25}{169}$.

(d) If $\vec{u}$ is a direction at which the temperature does not increase or decrease, then $D_{\vec{u}}T(3,4) = \nabla T(3,4)\cdot\vec{u} = 0$. This is equivalent to saying that $\vec{u}$ is perpendicular to $\nabla T(3,4)$. If $\vec{u} = u_1\vec{i}+u_2\vec{j}$ then we have $0 = -\frac{5}{169}\big(3\vec{i}+4\vec{j}\big)\cdot\vec{u} = -\frac{5}{169}(3u_1+4u_2)$. We may choose $\vec{u} = 4\vec{i}-3\vec{j}$. Therefore, the temperature does not change in the direction $4\vec{i}-3\vec{j}$.

(e) The unit vector in the direction $\langle 5,12\rangle$ is $\vec{u} = \frac{1}{13}\langle 5,12\rangle$. Thus the rate of change of T at $(3,4)$ in the direction $\langle 5,12\rangle$ is

$$\nabla T(3,4)\cdot\vec{u} = -\frac{5}{169}\big(3\vec{i}+4\vec{j}\big)\cdot\frac{1}{13}\big(5\vec{i}+12\vec{j}\big) = -\frac{5}{2187}(63) = -\frac{35}{243}.$$

(f) The unit vector in the direction $\langle 4,3\rangle$ is $\vec{u} = \frac{1}{5}\langle 4,3\rangle$. Thus the directional derivative of T at $(3,4)$ in the direction $\langle 4,3\rangle$ is

$$\nabla T(3,4)\cdot\vec{u} = -\frac{5}{169}\big(3\vec{i}+4\vec{j}\big)\cdot\frac{1}{5}\big(4\vec{i}+3\vec{j}\big) = -\frac{24}{169}.$$

□