Solution to Problem Set #6

1. Use Lagrange multipliers to find the maximum/minimum and maximizer/minimizer of $f$ subject to the given constraint.

(a) (50 point) $f(x, y) = xy, 4x^2 + y^2 = 1$

Solution. Let $f(x, y) = xy$ and $g(x, y) = 4x^2 + y^2$. The necessary conditions for the optimizer $(x, y)$ are
\[ \nabla f(x, y) = \lambda \nabla g(x, y) \]
and the constraint equations $4x^2 + y^2 = 1$ which are:
Since $\nabla f(x, y) = (y, x)$ and $\nabla g(x, y) = (8x, 2y)$, thus $(x, y)$ must satisfy
\[
\begin{align*}
0 & = 8\lambda x \\
0 & = 2\lambda y \\
4x^2 + y^2 & = 1
\end{align*}
\]

Method 1: From (1), (2) and (3), we know $\lambda \neq 0$ (b/c $\lambda = 0$ implies $x = 0$ and $y = 0$ which is impossible by $4x^2 + y^2 = 1$), $\lambda = \frac{y}{8x} = \frac{x}{2y}$.
This gives $2y^2 = 8x^2$ and $y = 2x$ or $y = -2x$. Using $4x^2 + y^2 = 1$, we have $(x, y) = (\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}}), (\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}}), (-\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}})$ or $(-\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}})$.
Recall $f(x, y) = xy$. Evaluate $f(0, 0) = 0$, $f(\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}}) = \frac{1}{4}$, $f(\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}}) = -\frac{1}{4}$, $f(-\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}}) = -\frac{1}{4}$ and $f(-\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}}) = \frac{1}{4}$. Thus the maximum is $\frac{1}{4}$, the minimum is $-\frac{1}{4}$, the maximizers are $(\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}}), (-\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}})$, and the minimizers are $(\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}}), (-\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}})$.

Method 2: From (1), (2), we get $xy = 8\lambda x^2, xy = 2\lambda y^2$. This gives $8\lambda x^2 = 2\lambda y^2, 8\lambda x^2 - 2\lambda y^2 = 2\lambda(2x - y)(2x + y) = 0$.
So $\lambda = 0$(impossible by the reason above) or $y = 2x$ or $y = -2x$.
Using $4x^2 + y^2 = 1$ and $y = \pm 2x$, we get $8x^2 = 1$ and $x = \pm \frac{1}{\sqrt{8}}$. So
$(x, y) = (\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}}), (\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}}), (-\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}})$ or $(-\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}})$.
Recall $f(x, y) = xy$. Evaluate $f(0, 0) = 0$, $f(\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}}) = \frac{1}{4}$, $f(\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}}) = -\frac{1}{4}$, $f(-\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}}) = -\frac{1}{4}$ and $f(-\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}}) = \frac{1}{4}$. Thus the maximum is $\frac{1}{4}$, the minimum is $-\frac{1}{4}$, the maximizers are $(\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}}), (-\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}})$, and the minimizers are $(\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}}), (-\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}})$.

\[\Box\]

(b) (50 point) $f(x, y, z) = 2x^2 - 2y + z^2, x^2 + y^2 + z^2 = 1$

Solution. Let $f(x, y, z) = 2x^2 - 2y + z^2, g(x, y, z) = x^2 + y^2 + z^2 - 1$. The necessary conditions for the optimizer $(x, y)$ are
\[ \nabla f(x, y, z) = \lambda \nabla g(x, y, z) \]
and the constraint equations which are:

\begin{align*}
(0.0.4) \quad 4x &= 2\lambda x \\
(0.0.5) \quad -2 &= 2\lambda y \\
(0.0.6) \quad 2z &= 2\lambda z \\
(0.0.7) \quad x^2 + y^2 + z^2 &= 1
\end{align*}

From (4), we have \( x = 0 \) or \( \lambda = 2 \).

Case 1 \( x = 0 \): We have \( y = -\frac{1}{\lambda} \) and \( \lambda \neq 0 \) from equation (5).

From equation (6), we have \( 2z - 2\lambda z = 2z(1 - \lambda) = 0 \). So \( z = 0 \) or \( \lambda = 1 \).

Case 1a \( z = 0 \): Since \( x = 0, z = 0 \) and \( x^2 + y^2 + z^2 = 1 \), we have \( y = \pm 1 \). So \( (x, y, z) = (0, 1, 0) \) or \( (0, -1, 0) \).

Case 1b \( \lambda = 1 \): From \( y = -\frac{1}{\lambda} \), we have \( y = -1 \). Since \( x = 0, y = -1 \) and \( x^2 + y^2 + z^2 = 1 \), we have \( z = 0 \). So \( (x, y, z) = (0, -1, 0) \).

Case 2 \( \lambda = 2 \): From equation (5), we have \( y = -\frac{1}{2} \). From equation (6), we have \( 2z = 4z \) and \( z = 0 \). Since \( z = 0, y = -\frac{1}{2} \) and \( x^2 + y^2 + z^2 = 1 \), we have \( x = \pm \frac{\sqrt{3}}{2} \). So \( (x, y, z) = (\pm \frac{\sqrt{3}}{2}, -\frac{1}{2}, 0) \).

The candidates for maximum or minimum are \( (0, \pm 1, 0) \) or \( (x, y, z) = (\pm \frac{\sqrt{3}}{2}, -\frac{1}{2}, 0) \). Recall that \( f(x, y, z) = 2x^2 - 2y + z^2 \), we have \( f(0, 1, 0) = -2 \), \( f(0, -1, 0) = 2 \), \( f(\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0) = 2 \) and \( f(\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0) = \frac{1}{2} \).

Thus the maximum is 2, the minimum is -2, the maximizers are \( (0, -1, 0) \) and \( (\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0) \). The minimizer is \( (0, 1, 0) \). \( \square \)