Solution to Problem Set #6

1. Use Lagrange multipliers to find the maximum/minimum and maximizer/minimizer of f subject to the given constraint.

(a) (50 point) $f(x, y) = xy$, $4x^2 + y^2 = 1$

Solution. Let $f(x, y) = xy$ and $g_(x, y) = 4x^2 + y^2$. The necessary conditions for the optimizer (x, y) are
$\nabla f(x, y) = \lambda \nabla g_(x, y)$ and the constraint equations $4x^2+y^2 = 1$ which are:
Since $\nabla f(x, y) = (y, x)$ and $\nabla g(x, y) = (8x, 2y)$, thus (x, y) must satisfy

(0.0.1) $$y = 8\lambda x$$
(0.0.2) $$x = 2\lambda y$$
(0.0.3) $$4x^2 + y^2 = 1$$

Method 1: From (1), (2) and (3), we know $\lambda \neq 0$ (b/c $\lambda = 0$ implies $x = 0$ and $y = 0$ which is impossible by $4x^2 + y^2 = 1$) , $\lambda = \frac{y}{8x} = \frac{x}{2y}$. This gives $2y^2 = 8x^2$ and $y = 2x$ or $y = -2x$. Using $4x^2 + y^2 = 1$, we have $(x, y) = (\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}})$, $(\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}})$, $(-\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}})$ or $(-\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}})$.
Recall $f(x, y) = xy$. Evaluate $f(0, 0) = 0$, $f(\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}}) = \frac{1}{4}$, $f(\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}}) = -\frac{1}{4}$, $f(-\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}}) = -\frac{1}{4}$ and $f(-\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}}) = \frac{1}{4}$. Thus the maximum is $\frac{1}{4}$, the minimum is $-\frac{1}{4}$, the maximizers are $(\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}})$, $(-\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}})$, and the minimizers are $(\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}})$, $(-\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}})$.

Method 2: From (1), (2), we get $xy = 8\lambda x^2$ $xy = 2\lambda y^2$. This gives $8\lambda x^2 = 2\lambda y^2$, $8\lambda x^2 - 2\lambda y^2 = 2\lambda(4x^2 - y^2) = 2\lambda(2x - y)(2x + y) = 0$. So $\lambda = 0$(impossible by the reason above) or $y = 2x$ or $y = -2x$. Using $4x^2 + y^2 = 1$ and $y = \pm 2x$, we get $8x^2 = 1$ and $x = \pm\frac{1}{\sqrt{8}}$. So $(x, y) = (\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}})$, $(\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}})$, $(-\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}})$ or $(-\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}})$.
Recall $f(x, y) = xy$. Evaluate $f(0, 0) = 0$, $f(\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}}) = \frac{1}{4}$, $f(\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}}) = -\frac{1}{4}$, $f(-\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}}) = -\frac{1}{4}$ and $f(-\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}}) = \frac{1}{4}$. Thus the maximum is $\frac{1}{4}$, the minimum is $-\frac{1}{4}$, the maximizers are $(\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}})$, $(-\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}})$, and the minimizers are $(\frac{1}{\sqrt{8}}, -\frac{2}{\sqrt{8}})$, $(-\frac{1}{\sqrt{8}}, \frac{2}{\sqrt{8}})$.

$\square$

(b) (50 point) $f(x, y, z) = 2x^2 - 2y + z^2$, $x^2 + y^2 + z^2 = 1$

Solution. Let $f(x, y, z) = 2x^2 - 2y + z^2$, $g_(x, y, z) = x^2 + y^2 + z^2 - 1$. The necessary conditions for the optimizer (x, y, z) are

$\nabla f(x,y,z) = \lambda \nabla g_(x,y,z)$ and the constraint equations which are:

(0.0.4) $$4x = 2\lambda x$$

(0.0.5) $$-2 = 2\lambda y$$

(0.0.6) $$2z = 2\lambda z$$

(0.0.7) $$x^2+y^2+z^2 = 1$$

From (4), we have $x = 0$ or $\lambda = 2$.

Case1 $x = 0$: We have $y = -\frac{1}{\lambda}$ and $\lambda \neq 0$ from equation (5).
From equation (6), we have $2z - 2\lambda z = 2z(1-\lambda) = 0$. So $z = 0$ or $\lambda = 1$.

Case 1a $z = 0$: Since $x = 0$, $z = 0$ and $x^2+y^2+z^2 = 1$, we have $y = \pm 1$. So $(x,y,z) = (0,1,0)$ or $(0,-1,0)$

Case 1b $\lambda = 1$: From $y = -\frac{1}{\lambda}$, we have $y = -1$. Since $x = 0$, $y = -1$ and $x^2+y^2+z^2 = 1$, we have $z = 0$. So $(x,y,z) = (0,-1,0)$.

Case 2 $\lambda = 2$: From equation (5), we have $y = -\frac{1}{2}$. From equation (6), we have $2z = 4z$ and $z = 0$. Since Since $z = 0$, $y = -\frac{1}{2}$ and $x^2+y^2+z^2 = 1$, we have $x = \pm\frac{\sqrt{3}}{2}$. So $(x,y,z) = (\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0)$.

The candidates for maximum or minimum are $(0,\pm 1,0)$ or $(x,y,z) = (\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0)$. Recall that $f(x,y,z) = 2x^2 - 2y + z^2$, we have $f(0,1,0) = -2$, $f(0,-1,0) = 2$, $f(\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0) = 2$ and $f(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0) = \frac{1}{2}$.
Thus the maximum is 2, the minimum is -2, the maximizers are $(0,-1,0)$ and $(\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0)$. The minimizer is $(0,1,0)$. $\square$