Solution to Problem Set #7

1. **(a)** (5 pt) If $f(x,y)$ gives the pollution density, in micrograms per square meter, and x and y are measured in meters, give the units and practical interpretation of $\int\int_R f(x,y)\,dA$.

Solution. The unit of $\int\int_R f(x,y)\ dA$ is $\frac{micrograms}{meter^2} \cdot meter \cdot meter = micrograms$. The meaning of $\int\int_R f(x,y)\ dA$ is the total amount of pollution is the region A.

□

(b) (5 pt) Using Riemann sums with two subdivisions in each direction, find upper and lower bounds for the volume under the graph of $f(x,y) = 2 + xy$ above the rectangle R with $0 \le x \le 2$, $0 \le y \le 2$.

Solution. Let $f(x,y) = 2 + xy$. We have $f_x = y \ge 0$ and $f_y = x \ge 0$ in the rectangle R. The lower estimate is $f(0,0)\cdot 1 + f(1,0)\cdot 1 + f(0,1)\cdot 1 + f(1,1)\cdot 1 = 2+2+2+3 = 9$.
The upper estimate is $f(1,1)\cdot 1 + f(1,2)\cdot 1 + f(2,1)\cdot 1 + f(2,2)\cdot 1 = 3+4+4+6 = 17$. □

2. (12 pt) Find the volume of the solid bounded by the surface $z = y\sqrt{y^2+x}$ and the planes $x=0$, $x=1$, $y=0$, $y=1$ and $z=0$.

Solution. The volume is $\int_0^1\int_0^1 y\sqrt{y^2+x}dydx$ Let $u = y^2+x$. Then $du = 2ydy$, $ydy = \frac{du}{2}$ and $\int y\sqrt{y^2+x}dy = \int \frac{u^{1/2}}{2}du = \frac{u^{3/2}}{3} + C = \frac{(y^2+x)^{3/2}}{3} + C$.

So $\int_0^1\int_0^1 y\sqrt{y^2+x}dydx = \int_0^1 \frac{(y^2+x)^{3/2}}{3}\Big|_0^1 dx$

$= \int_0^1 \frac{(1+x)^{3/2}}{3} - \frac{(x)^{3/2}}{3}dx = \frac{2(1+x)^{5/2}}{15} - \frac{2(x)^{5/2}}{15}\Big|_0^1 = \frac{2(2)^{5/2}}{15} - \frac{2}{15} - (\frac{2}{15} - 0)$

$= \frac{2(2)^{5/2}}{15} - \frac{4}{15} = \frac{8\sqrt{2}}{15} - \frac{4}{15}$ □

3. Compute the following iterated integrals.

(a) (12 pt) $\int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}dydx$

Solution. $\int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}dydx$.
Let $u = x^2+y^2+1$. Then $du = 2ydy$, $ydy = \frac{du}{2}$ and $\int \frac{xy}{\sqrt{x^2+y^2+1}}dy = \int \frac{x}{2\sqrt{u}}du = \int \frac{xu^{-\frac{1}{2}}}{2}du = xu^{\frac{1}{2}} = x(x^2+y^2+1)^{\frac{1}{2}}$. Hence $\int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}dydx = \int_0^1 x(x^2+y^2+1)^{\frac{1}{2}}\Big|_0^1 dx = \int_0^1 (x(x^2+2)^{\frac{1}{2}} - x(x^2+1)^{\frac{1}{2}})dx$
$= (use\ substitution\ u = x^2+2)(\frac{(x^2+2)^{\frac{3}{2}}}{3} - \frac{(x^2+1)^{\frac{3}{2}}}{3})\Big|_0^1 = (\frac{(3)^{\frac{3}{2}}}{3} - \frac{(2)^{\frac{3}{2}}}{3}) - (\frac{(2)^{\frac{3}{2}}}{3} - \frac{1}{3}) = \sqrt{3} - \frac{4\sqrt{2}}{3} + \frac{1}{3}$. □

(b) (12 pt)$\int_0^1 \int_0^4 e^y\sqrt{x+e^y}dxdy$

Solution. $\int_0^1 \int_0^4 e^y\sqrt{x+e^y}dxdy$.
Let $u = x+e^y$. Then $du = dx$ and $\int e^y\sqrt{x+e^y}dx = \int e^y\sqrt{u}du = \int e^y u^{1/2}du = \frac{2e^y u^{3/2}}{3} = \frac{2e^y(x+e^y)^{3/2}}{3}$.
Hence $\int_0^1 \int_0^4 e^y\sqrt{x+e^y}dxdy = \int_0^1 \int_0^4 e^y\sqrt{x+e^y}dxdy = \int_0^1 \frac{2e^y(x+e^y)^{3/2}}{3}\Big|_0^4 dy$ $\int_0^1 \frac{2e^y(4+e^y)^{3/2}}{3} - \frac{2e^y(e^y)^{3/2}}{3}dy = \int_0^1 \frac{2e^y(4+e^y)^{3/2}}{3} - \frac{2e^{\frac{5y}{2}}}{3}dy = \frac{4(4+e^y)^{5/2}}{15} - \frac{4e^{\frac{5y}{2}}}{15}\Big|_0^1 =$ $(\frac{4(4+e)^{5/2}}{15} - \frac{4e^{\frac{5}{2}}}{15}) - (\frac{4(5)^{5/2}}{15} - \frac{4}{15}) = \frac{4(4+e)^{5/2}}{15} - \frac{4e^{\frac{5}{2}}}{15} - \frac{4(5)^{5/2}}{15} + \frac{4}{15}$. □

4. (12 pt) Evaluate $\int\int_D \frac{4y}{x^3+4}dA$ where $D = \{(x,y)|1 \le x \le 6, 0 \le y \le 4x\}$.

Solution. $\int\int_D \frac{4y}{x^3+4}dA = \int_1^6 \int_0^{4x} \frac{4y}{x^3+4}dydx = \int_1^6 \frac{2y^2}{x^3+4}\Big|_0^{4x}dx = \int_1^6 \frac{32x^2}{x^3+4}dx$. Let $u = x^3+4$. Then $du = 3x^2dx$, $x^2dx = \frac{u}{3}du$ and $\int \frac{32x^2}{x^3+4}dx = \int \frac{32}{3u}du = \frac{32}{3}\ln|u| + C = \frac{32}{3}\ln|x^3+4| + C$. Hence $\int\int_D \frac{4y}{x^3+4}dA = \frac{32}{3}\ln|x^3+4|\Big|_1^6 = \frac{32}{3}\ln(220) - \frac{32}{3}\ln(5)$.

□

5. (12 pt)Evaluate $\int\int_D x\sqrt{y^2-x^2}dA$ where $D = \{(x,y)|0 \le x \le y, 0 \le y \le 3\}$

Solution. $\int\int_D x\sqrt{y^2-x^2}dA = \int_0^3 \int_0^y x\sqrt{y^2-x^2}dxdy = \int_0^3 \int_0^y x(y^2-x^2)^{\frac{1}{2}}dxdy$.
Let $u = y^2-x^2$. Then $du = -2xdx$, $xdx = -\frac{u}{2}du$ and $\int -\frac{(u)^{\frac{1}{2}}}{2}du = \int -\frac{1}{3}u^{\frac{3}{2}} = -\frac{1}{3}(y^2-x^2)^{\frac{3}{2}}$. Hence $\int_0^3 \int_0^y x(y^2-x^2)^{\frac{1}{2}}dxdy = \int_0^3 -\frac{1}{3}(y^2-x^2)^{\frac{3}{2}}\Big|_0^y dy =$ $\int_0^3 \frac{1}{3}(y^2)^{\frac{3}{2}}dy = \int_0^3 \frac{1}{3}y^3dy = \frac{1}{12}y^4\Big|_0^3 = \frac{81}{12} = \frac{27}{4}$. □

6. (15 pts) $\int_0^1 \int_y^1 e^{x^2}dxdy$

Solution. Let $D = \{(x,y)|y \le x \le 1, 0 \le y \le 1\}$ Then $0 \le y \le x$ and $0 \le x \le 1$. So D is the same as $\{(x,y)|0 \le x \le 1, 0 \le y \le x\}$.
We have $\int_0^1 \int_y^1 e^{x^2}dxdy = \int_0^1 \int_0^x e^{x^2}dydx = \int_0^1 e^{x^2}y\Big|_0^x dx = \int_0^1 e^{x^2}xdx = \frac{e^{x^2}}{2}|_0^1 = \frac{e}{2} - \frac{1}{2}$.

□

7. (15 pts) $\int_0^1 \int_{x^2}^1 x^3\sin(3y^3)dydx$

Solution. Let $D = \{(x,y)|0 \le x \le 1, x^2 \le y \le 1\}$. Since $x^2 \le y$, we have $x \le \sqrt{y}$, $0 \le x \le \sqrt{y}$ and $0 \le y \le 1$. So D is the same as $\{(x,y)|0 \le x \le \sqrt{y}, 0 \le y \le 1\}$.

We have $\int_0^1 \int_{x^2}^1 x^3 \sin(3y^3) dy dx = \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(3y^3) dx dy = \int_0^1 \frac{x^4}{4} \sin(3y^3) \Big|_0^{\sqrt{y}} dx = \int_0^1 \frac{y^2}{4} \sin(3y^3) dx = \frac{-\cos(3y^3)}{36} \Big|_0^1 = \frac{-\cos(3)}{36} + \frac{1}{36}$.

$\square$