Solution to Problem Set #8

1. (20 pt) Find the volume of an ice cream cone bounded by the hemisphere $z = \sqrt{8 - x^2 - y^2}$ and the cone $z = \sqrt{x^2 + y^2}$. The graphs above are the graphs of $z = \sqrt{8 - x^2 - y^2}$, $z = \sqrt{x^2 + y^2}$ and their intersection.

Solution.
The region is bounded above by the hemisphere \( z = \sqrt{8 - x^2 - y^2} \) and below by the cone \( z = \sqrt{x^2 + y^2} \). We have \( \sqrt{x^2 + y^2} \leq z \leq \sqrt{8 - x^2 - y^2} \).

Thus \( x^2 + y^2 \leq z^2 \leq 8 - x^2 - y^2 \) and \( x^2 + y^2 \leq 4 \).

In polar coordinates, this region \( x^2 + y^2 \leq 4 \) is \( R = \{(r, \theta) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\} \). Note that \( \sqrt{8 - x^2 - y^2} = \sqrt{8 - r^2} \) and \( \sqrt{x^2 + y^2} = r \).

Hence, we can compute the volume of the ice cream cone by finding the volume under the graph of \( \sqrt{8 - r^2} \) above the disk \( R \) and subtracting the volume under the graph of \( r \) above \( R \). Therefore, we have

\[
\text{Volume} = \int_0^{2\pi} \int_0^2 r \sqrt{8 - r^2} \,dr \, d\theta - \int_0^{2\pi} \int_0^2 \sqrt{r^2} \,dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^2 r \sqrt{8 - r^2} - r^2 \,dr \, d\theta
= \int_0^{2\pi} \left[ -\frac{1}{3}(8 - r^2)^{3/2} - \frac{1}{3}r^3 \right]_0^2 \,d\theta
= \frac{1}{3} \int_0^{2\pi} \left( -4^{3/2} - 8 + 8^{3/2} \right) \,d\theta
= \frac{1}{3} (16\sqrt{2} - 16) \int_0^{2\pi} \,d\theta
= \frac{1}{3} (32\pi)(\sqrt{2} - 1). \quad \Box
\]

2. Evaluate the following integral by converting to polar coordinates.

(a) (10 pt) \( \int_0^2 \int_0^{\sqrt{4-y^2}} (x^2 + y^2)^{3/2} \,dxdy \)

(b) (10 pt) \( \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sin(x^2 + y^2) \,dy \,dx \)
Solution. (a) The region of integration is \( \{(x, y) | 0 \leq x \leq \sqrt{4-y^2}, 0 \leq y \leq y\} \). The is the region in first quadrant. In polar coordinates, it is \( R = \{(r, \theta) : 0 \leq r \leq 2, 0 \leq \theta \leq \frac{\pi}{2}\} \). We also have \( (x^2 + y^2)^{\frac{3}{2}} = (r^2)^{\frac{3}{2}} = r^3 \) and
\[
\int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 + y^2 \, dydx = \int_{0}^{2\pi} \int_{0}^{1} \sin(r^2) \cdot r \, drd\theta
\]
\[
= \int_{0}^{2\pi} \left( -\cos(1) + \frac{1}{2} \right) \, d\theta = \left( -\cos(1) + \frac{1}{2} \right) \cdot 2\pi = -\cos(1) + 1.
\]
(b) The region of integration is \( \{(x, y) | -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, -1 \leq x \leq 1\} \). Note that \(-\sqrt{1-x^2} = y \) and \(\sqrt{1-x^2} = y \) imply \(x^2 + y^2 = 1\). Since \(-1 \leq x \leq 1\), we know that \( R \) is the disk inside the unit circle.

In polar coordinates, it is \( R = \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\} \). We also have \( x^2 + y^2 = r^2 \) and
\[
\int_{0}^{2\pi} \int_{0}^{1} (x^2 + y^2)^{\frac{3}{2}} \, drd\theta = \int_{0}^{2\pi} \int_{0}^{1} r^3 \cdot r \, drd\theta
\]
\[
= \int_{0}^{2\pi} \int_{0}^{1} r^4 \, drd\theta = \int_{0}^{2\pi} \frac{32}{3} \, d\theta = \frac{32}{3} \cdot \frac{\pi}{2} = \frac{16\pi}{3}.
\]

3. (a) (10 pt) For \( a > 0 \) find the volume under the graph of \( z = e^{-(x^2+y^2)} \) above the disk \( x^2 + y^2 \leq a^2 \).
(b) (10 pt) What happens to the volume as \( a \to \infty \).

Solution. In polar coordinates, the disk is described by the inequalities \( 0 \leq r \leq a, 0 \leq \theta \leq 2\pi \) and the function is \( e^{-r^2} \). Hence, the volume under the graph and above the disk is
\[
\text{Volume} = \int_{0}^{a} \int_{0}^{2\pi} e^{-r^2} \, rdr \, d\theta = 2\pi \int_{0}^{a} re^{-r^2} \, dr = 2\pi \left[ -\frac{1}{2} e^{-r^2} \right]_{0}^{a} = \pi (1 - e^{-a^2}).
\]
Since \( \lim_{a \to \infty} \pi (1 - e^{-a^2}) = \pi \), the volume under the graph of \( e^{-x^2-y^2} \) above the entire xy-plane is \( \pi \).

4. Consider a thin plate that occupies the region \( D \) bounded by the parabola \( y = 1 - x^2 \), \( x = 0 \) and \( y = 0 \) in the first quadrant with density function \( \rho(x, y) = x \).
(a) (10 pt) Find the mass of the thin plate.
(b) (10 pt) Find the center of mass of the thin plate.
(c) (10 pt) Find moments of inertia \( I_x, I_y \) and \( I_0 \).

Solution. (a) The graph \( y = 1 - x^2 \) intersect with \( y = 0 \) at \( 1 - x^2 = 0 \), i.e. \( x = \pm 1 \). We also know that \( y = 1 - x^2 \geq 0 \) when \(-1 \leq x \leq 1\). The region of integration is \( R = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1 - x^2\} \).

The mass is \( m = \int \int_{R} \rho(x, y) \, dA = \int_{0}^{1} \int_{0}^{1-x^2} x \, dxdy = \int_{0}^{1} \int_{0}^{1-x^2} xy |_{0}^{1-x^2} \, dx = \int_{0}^{1} x(1-x^2) \, dx = \int_{0}^{1} (x - x^3) \, dx = \left( \frac{x^2}{2} - \frac{x^4}{4} \right) |_{0}^{1} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \).

(b) The center of mass is \( \left( \frac{\int \int_{R} x \rho(x, y) \, dA}{m}, \frac{\int \int_{R} y \rho(x, y) \, dA}{m} \right) \).
Now \( \int \int_{R} \rho(x,y)x \, dA = \int_{0}^{1} \int_{0}^{1-x^{2}} x \, dy \, dx = \int_{0}^{1} x^{2}(1-x^{2}) \, dx = \int_{0}^{1} (x^{2} - x^{4}) \, dx = \left[ \frac{x^{3}}{3} - \frac{x^{5}}{5} \right]_{0}^{1} = \frac{1}{3} - \frac{1}{5} = \frac{2}{15} \).

\( \int \int_{R} \rho(x,y)y \, dA = \int_{0}^{1} \int_{0}^{1-x^{2}} xy \, dy \, dx = \int_{0}^{1} \frac{1}{2}(1-x^{2}) \, dx = \int_{0}^{1} \frac{1}{2}x^{2} - \frac{1}{2}x^{4} \, dx = \left[ \frac{x^{3}}{3} - \frac{x^{5}}{3} \right]_{0}^{1} = \frac{1}{3} - \frac{1}{5} = \frac{2}{15} \).

So the center of mass is \( \left( \frac{2}{15}, \frac{2}{15} \right) \).

\( \frac{\int \int_{R} \rho(x,y)y^{2} \, dA}{\int \int_{R} \rho(x,y) \, dA} = \int_{0}^{1} \int_{0}^{1-x^{2}} y^{2} \, dy \, dx = \int_{0}^{1} \left[ \frac{1}{3}x^{3} - \frac{1}{5}x^{5} \right]_{0}^{1} = \frac{1}{3} - \frac{1}{5} = \frac{2}{15} \).

Let \( u = 1 - x^{2} \). Then \( du = -2x \, dx \), \( x \, dx = -\frac{du}{2} \) and \( \int x^{1-x^{2}} \, dx = -\int \frac{u^{4}}{2} \, du = -\frac{u^{5}}{24} = -\frac{(1-x^{2})^{4}}{24} \).

So \( I_{x} = -\frac{(1-x^{2})^{4}}{24} \bigg|_{0}^{1} = \frac{1}{24} \).

\( I_{y} = \int \int_{R} \rho(x,y)x^{2} \, dA = \int_{0}^{1} \int_{0}^{1-x^{2}} x^{2} \, dy \, dx = \int_{0}^{1} \left[ \frac{1}{3}x^{3} - \frac{1}{5}x^{5} \right]_{0}^{1} = \frac{1}{3} - \frac{1}{5} = \frac{2}{15} \).

Note that \( I_{0} = \int \int_{R} \rho(x,y)(x^{2} + y^{2}) \, dA = I_{x} + I_{y} = \frac{1}{20} + \frac{1}{24} = \frac{11}{120} \).