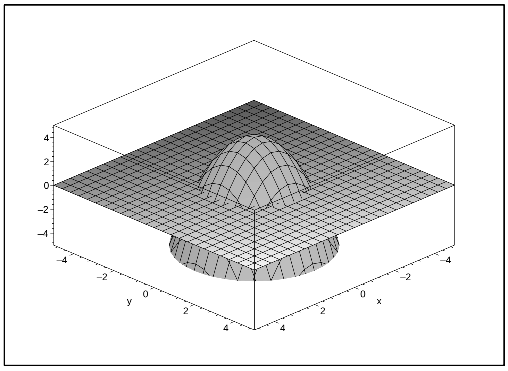
Solution to Problem Set #9

1. Find the area of the following surface.

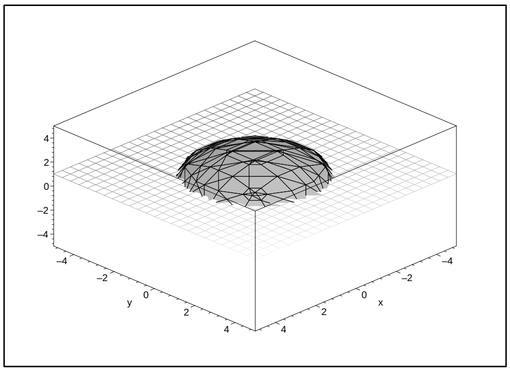
(a) (15 pts) The part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the $x - y$ plane.



Solution. The part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the $x - y$ plane must satisfy $z = 9 - x^2 - y^2 \geq 0$. Thus $x^2 + y^2 \leq 9$. We have $z = f(x, y) = 9 - x^2 - y^2$, $f_x = -2x$, $f_y = -2y$ and $\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + (-2x)^2 + (-2y)^2} = \sqrt{1 + 4x^2 + 4y^2}$.
The region $E = \{(x, y)|x^2 + y^2 \leq 9\}$ is $\{(r, \theta)|0 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$ in polar coordinates. Hence the area of surface is
$\int\int_E \sqrt{1 + f_x^2 + f_y^2}dxdy = \int_0^{2\pi}\int_0^3 \sqrt{1 + r^2}rdrd\theta = \int_0^{2\pi} \frac{1}{3}(1 + r^2)^{\frac{3}{2}}|_0^3 d\theta = (\frac{1}{3}(10)^{\frac{3}{2}} - \frac{1}{3}) \cdot 2\pi = \frac{2\pi}{3}((10)^{\frac{3}{2}} - 1)$.

□

(b) (15 pts)The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$.



Solution. The sphere $x^2+y^2+z^2 = 4$ can be written as $z = \sqrt{4 - x^2 - y^2}$. The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$ must satisfy $1 \leq z = \sqrt{4 - x^2 - y^2}$. Thus $x^2 + y^2 \leq 3$. We have $z = f(x, y) = \sqrt{4 - x^2 - y^2}$, $f_x = \frac{-x}{\sqrt{4-x^2-y^2}}$, $f_y = \frac{-y}{\sqrt{4-x^2-y^2}}$ and

$\sqrt{1+f_x^2+f_y^2} = \sqrt{1+(\frac{-x}{\sqrt{4-x^2-y^2}})^2+(\frac{-y}{\sqrt{4-x^2-y^2}})^2} = \sqrt{1+\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}} = \sqrt{\frac{4-x^2-y^2+x^2+y^2}{4-x^2-y^2}}\sqrt{\frac{4}{4-x^2-y^2}} = 2\sqrt{\frac{1}{4-x^2-y^2}}$.

The region $E = \{(x,y)|x^2+y^2 \le 3\}$ is $\{(r,\theta)|0 \le r \le \sqrt{3}, 0 \le \theta \le 2\pi\}$ in polar coordinates. Hence the area of surface is $\int\int_E \sqrt{1+f_x^2+f_y^2}dxdy = \int_0^{2\pi}\int_0^{\sqrt{3}} 2\sqrt{\frac{1}{4-r^2}}rdrd\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} 2(4-r^2)^{-\frac{1}{2}}rdrd\theta = \int_0^{2\pi} -2(4-r^2)^{\frac{1}{2}}|_0^{\sqrt{3}}d\theta = (-2+4)\cdot 2\pi = 4\pi$.

□

2. Evaluate the following triple integrals:

(a) (15 pts)$\int\int\int_E yz\sin(x^5)dV$ where E is the region $\{(x,y,z)|0 \le x \le 1, 0 \le y \le x, x \le z \le 2x\}$

Solution. The region E bounded by the xy, yz, xz planes and the plane $2x+y+2z=4$ is the set $\{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le 2, 0 \le y \le 4-2x, 0 \le z \le \frac{1}{2}(4-x-2y)\}$.

$$\int\int\int_E zdV = \int_0^2\int_0^{4-2x}\int_0^{\frac{1}{2}(4-x-2y)} zdz\ dy\ dx = \int_0^2\int_0^{4-2x} \frac{1}{2}z^2\Big|_0^{\frac{1}{2}(4-x-2y)} dy\ dx$$

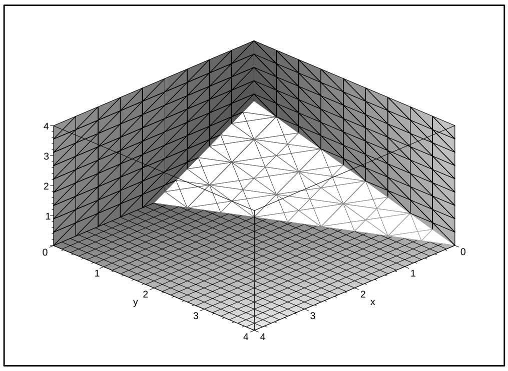
$$= \int_0^2\int_0^{4-2x} \tfrac{1}{8}(4-x-2y)^2\ dy\ dx = \int_0^2 -\tfrac{1}{48}(4-x-2y)^3\Big|_0^{4-2x} dx \text{ (by substitution u=4-x-2y)}$$

$$= \int_0^2 -\tfrac{1}{48}(4-x-2(4-2x))^3 + \tfrac{1}{48}(4-x)^3\ dx = \int_0^2 -\tfrac{1}{48}(-4+3x)^3 + \tfrac{1}{48}(4-x)^3\ dx$$

$$= -\tfrac{1}{48\cdot3\cdot4}(-4+3x)^4 - \tfrac{1}{48\cdot4}(4-x)^4\Big|_0^2 = -\tfrac{1}{48\cdot3\cdot4}16 - \tfrac{1}{48\cdot4}16 - (-\tfrac{1}{48\cdot3\cdot4}256 - \tfrac{1}{48\cdot4}256) = -\tfrac{1}{36} - \tfrac{1}{12} + \tfrac{4}{9} + \tfrac{4}{3} = \frac{5}{3}.$$

□

(b) (15 pts)$\int\int\int_E zdV$ where E is the region bounded by $x=0$, $y=0$, $z=0$ and $2x+y+2z=4$.



Solution. The region E bounded by the xy, yz, xz planes and the plane $2x + y + 2z = 4$ is the set $\{(x, y, z) \in \mathbb{R}^3 : 0 \le x \le 2, 0 \le y \le 4 - 2x, 0 \le z \le \frac{1}{2}(4 - x - 2y)\}$.

$$\int\int\int_E z dV = \int_0^2 \int_0^{4-2x} \int_0^{\frac{1}{2}(4-x-2y)} z dz\, dy\, dx = \int_0^2 \int_0^{4-2x} \frac{1}{2}z^2\Big|_0^{\frac{1}{2}(4-x-2y)} dy\, dx$$

$$= \int_0^2 \int_0^{4-2x} \tfrac{1}{8}(4-x-2y)^2\, dy\, dx = \int_0^2 -\tfrac{1}{48}(4-x-2y)^3\Big|_0^{4-2x} dx \text{ (by substitution u=4-x-2y)}$$

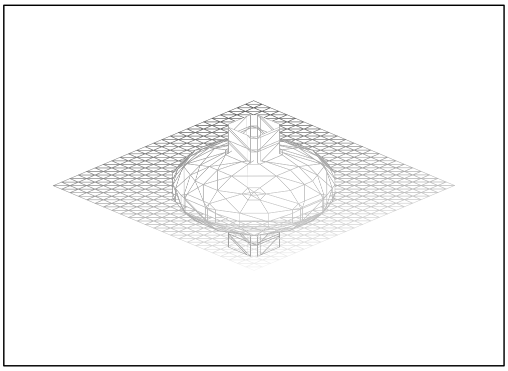
$$= \int_0^2 -\tfrac{1}{48}(4 - x - 2(4 - 2x))^3 + \tfrac{1}{48}(4 - x)^3\, dx = \int_0^2 -\tfrac{1}{48}(-4 + 3x)^3 + \tfrac{1}{48}(4 - x)^3\, dx$$

$$= -\tfrac{1}{48\cdot3\cdot4}(-4 + 3x)^4 - \tfrac{1}{48\cdot4}(4 - x)^4\Big|_0^2 = -\tfrac{1}{48\cdot3\cdot4}16 - \tfrac{1}{48\cdot4}16 - (-\tfrac{1}{48\cdot3\cdot4}256 - \tfrac{1}{48\cdot4}256)$$

$$= -\tfrac{1}{36} - \tfrac{1}{12} + \tfrac{4}{9} + \tfrac{4}{3} = \frac{5}{3}.$$

□

3. (10 pts)A bead is made by drilling a cylindrical hole of radius 1 mm through a sphere of radius 9 mm Set up a triple integral in cylindrical coordinates representing the volume of the bead. Evaluate the integral. (Hint: Express the region $E = \{(x, y, z)|x^2+y^2+z^2 \le 9$ and $x^2+y^2 \ge 1\}$ (There is a typo in the original problem.) in cylindrical coordinates and find $\int\int\int_E dV$.)



Solution. In cylindrical coordinates, the sphere is given by the equation $r^2 + z^2 = 9$ and the hole is given by $r = 1$. Hence, the bead is described by the inequalities $-\sqrt{9 - r^2} \le z \le \sqrt{9 - r^2}$, $0 \le \theta \le 2\pi$ and $1 \le r \le 3$. We find the volume by integrating the constant density function 1 over the sphere:

$$\text{Volume} = \int_R 1\ dV = \int_1^3 \int_0^{2\pi} \int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} r\ dz\ d\theta\ dr = \int_1^3 \int_0^{2\pi} \Big[rz\Big]_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}}\ d\theta\ dr$$
$$= \int_1^3 2r\sqrt{9-r^2} \int_0^{2\pi} d\theta\ dr = 2\pi\big[-\tfrac{2}{3}(9-r^2)^{3/2}\big]_1^3$$
$$= \tfrac{2}{3}(8)^{3/2}2\pi = \tfrac{4\pi}{3}8\sqrt{8} = \tfrac{64\pi}{3}\sqrt{2}\ \text{mm}^3\,. \qquad \square$$

4. (a) (15 pts)A spherical cloud of gas of radius 3 km is more dense at the center than toward the edge. At a distance of ρ km from the center, the density is $\delta(\rho) = 3 - \rho$. Write an integral representing the total mass of the cloud of gas and evaluate it.

Solution. In spherical coordinates, the sphere is described by the inequalities $0 \le \rho \le 3$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$. Hence, the total mass of the cloud is

$$\text{Mass} = \int_W \delta\ dV = \int_0^{\pi}\int_0^{2\pi}\int_0^3 (3-\rho)\ \rho^2 \sin(\phi)\ d\rho\ d\theta\ d\phi$$
$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin(\phi)\ d\phi\right)\left(\int_0^3 3\rho^2 - \rho^3\ d\rho\right)$$
$$= (2\pi)\left[-\cos(\phi)\right]_0^{\pi}\left[\rho^3 - \tfrac{1}{3}\rho^4\right]_0^3 = (2\pi)(2)\left(27\left[1-\tfrac{3}{4}\right]\right) = 27\pi\,. \qquad \square$$

(b) (15 pts)A half-melon is approximated by the region between two concentric spheres, one a radius 1 and the other of radius 2. Write a triple integral, including limits of integration, giving the volume of the half-melon. Evaluate the integral.

Solution. In spherical coordinates, the outer hemisphere is described by the inequalities $0 \le \rho \le b$, $0 \le \phi \le \pi/2$ and $0 \le \theta \le 2\pi$ and the inner hemisphere is described by $0 \le \rho \le a$, $0 \le \phi \le \pi/2$ and $0 \le \theta \le 2\pi$. Hence, the volume of the half-melon is

$$\text{Volume} = \int_W 1\ dV = \int_0^{2\pi}\int_0^{\pi/2}\int_1^2 \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$$
$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/2} \sin(\phi)\ d\phi\right)\left(\int_1^2 \rho^2\ d\rho\right)$$
$$= (2\pi)\left[-\cos(\phi)\right]_0^{\pi/2}\left[\tfrac{1}{3}\rho^3\right]_1^2 = \tfrac{2\pi}{3}(2^3 - 1^3) = \tfrac{14\pi}{3}\,. \qquad \square$$