1. Use Lagrange multipliers to find the maximum/minimum and maximizer/minimizer of \( f \) subject to the given constraint.

\[ f(x, y) = xy, \quad x^2 + y^2 = 1 \]

**Solution.** Let \( f(x, y) = xy \) and \( g(x, y) = x^2 + y^2 \). The necessary conditions for the optimizer \((x, y)\) are

\[
\nabla f(x, y) = \lambda \nabla g(x, y)
\]

and the constraint equations \( x^2 + y^2 = 1 \) which are:

Since \( \nabla f(x, y) = (y, x) \) and \( \nabla g(x, y) = (2x, 2y) \), thus \((x, y)\) must satisfy

\[
\begin{align*}
    y &= 2\lambda x \quad (0.0.1) \\
    x &= 2\lambda y \quad (0.0.2) \\
    x^2 + y^2 &= 1 \quad (0.0.3)
\end{align*}
\]

From (1), (2), we get \( xy = \lambda x^2 \) \( xy = 2\lambda y^2 \). This gives \( 2\lambda x^2 = 2\lambda y^2 \), \( 2\lambda x^2 - 2\lambda y^2 = 2\lambda(x^2 - y^2) = 2\lambda(x - y)(x + y) = 0 \). So \( \lambda = 0 \) (impossible b/c this implies \( x = 0 \) and \( y = 0 \)) or \( y = x \) or \( y = -x \). Using \( x^2 + y^2 = 1 \) and \( y = \pm x \), we get \( 2x^2 = 1 \) and \( x = \pm \frac{1}{\sqrt{2}} \). So \((x, y) = (\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{2}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, -\frac{2}{\sqrt{2}}) \) or \((\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}})\).

Recall \( f(x, y) = xy \). \( f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{2}, \enspace f(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = -\frac{1}{2}, \enspace f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = -\frac{1}{2} \) and \( f(-\frac{1}{\sqrt{1}}, \frac{1}{\sqrt{1}}) = \frac{1}{2} \). Thus the maximum is \( \frac{1}{4} \), the minimum is \( -\frac{1}{2} \), the maximizers are \((\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\), and the minimizers are \((\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\).

\( \square \)

2. \[ \int_0^1 \int_x^1 e^{y^2} dy dx. \]

**Solution.** Let \( D = \{(x, y)|0 \leq x \leq 1, x \leq y \leq 1\} \) Then \( 0 \leq x \leq y \) and \( 0 \leq x \leq y \leq 1 \) \( D \) is the same as \( \{(x, y)|0 \leq x \leq y, 0 \leq y \leq 1\} \).

We have \( \int_0^1 \int_x^1 e^{y^2} dy dx = \int_0^1 \int_0^y e^{y^2} dx dy = \int_0^1 \int_0^y xe^{y^2}|_0^y dx = \int_0^1 e^{y^2} y dx = \frac{e^{y^2}}{2} |_0^1 = \frac{e}{2} - \frac{1}{2} \).

\( \square \)