MATH 2850 Solution to Quiz #6

1. Use Lagrange multipliers to find the maximum/minimum and maximizer/minimizer of f subject to the given constraint.
$f(x, y) = xy$, $x^2 + y^2 = 1$

Solution. Let $f(x, y) = xy$ and $g_(x, y) = x^2+y^2$. The necessary conditions for the optimizer (x, y) are
$\nabla f(x, y) = \lambda \nabla g_(x, y)$ and the constraint equations $x^2+y^2 = 1$ which are:
Since $\nabla f(x, y) = (y, x)$ and $\nabla g(x, y) = (2x, 2y)$, thus (x, y) must satisfy

$$y = 2\lambda x \tag{0.0.1}$$
$$x = 2\lambda y \tag{0.0.2}$$
$$x^2 + y^2 = 1 \tag{0.0.3}$$

From (1), (2), we get $xy = \lambda x^2$ $xy = 2\lambda y^2$. This gives $2\lambda x^2 = 2\lambda y^2$, $2\lambda x^2 - 2\lambda y^2 = 2\lambda(x^2 - y^2) = 2\lambda(x - y)(x + y) = 0$. So $\lambda = 0$(impossible b/c This implies $x = 0$ and $y = 0$) or $y = x$ or $y = -x$. Using $x^2 + y^2 = 1$ and $y = \pm x$, we get $2x^2 = 1$ and $x = \pm\frac{1}{\sqrt{2}}$. So $(x, y) = (\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, -\frac{2}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{2}{\sqrt{2}})$ or $(-\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}})$.
Recall $f(x, y) = xy$. $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{2}$, $f(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = -\frac{1}{2}$, $f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{1}}) = -\frac{1}{2}$ and $f(-\frac{1}{\sqrt{1}}, \frac{1}{\sqrt{1}}) = \frac{1}{2}$. Thus the maximum is $\frac{1}{4}$, the minimum is $-\frac{1}{2}$, the maximizers are $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, and the minimizers are $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{1}})$.

□

2. $\int_0^1 \int_x^1 e^{y^2} dydx$.

Solution. Let $D = \{(x, y)|0 \le x \le 1, x \le y \le 1\}$ Then $0 \le x \le y$ and $0 \le x \le y \le 1$. So D is the same as $\{(x, y)|0 \le x \le y, 0 \le y \le 1\}$.
We have $\int_0^1 \int_x^1 e^{y^2} dydx = \int_0^1 \int_0^y e^{y^2} dxdy = \int_0^1 \int_0^y xe^{y^2}|_0^y dx = \int_0^1 e^{y^2} ydx == \frac{e^{y^2}}{2}|_0^1 = \frac{e}{2} - \frac{1}{2}$.

□