Solution to Review Problems for Final Exam

MATH 3860

Please email(mao-pei.tsui@utoledo.edu) me if you find any mistake.

Here is the directory of the problems.

(1) (a) Let $v = ax + by + c$. Since $\frac{dv}{dx} = a + b\frac{dy}{dx}$ and $\frac{dy}{dx} = F(ax + by + c)$, we have $\frac{dv}{dx} = a + bF(ax + by + c) = a + bF(v)$. Thus $\int \frac{dv}{a+bF(v)} = \int dx$.

(b) Let $v = \frac{y}{x}$, i.e. $y = xv$. Using $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and $\frac{dy}{dx} = F(\frac{y}{x}) = F(v)$, we have $v + x\frac{dv}{dx} = F(v)$. It can be rewritten as $x\frac{dv}{dx} = F(v) - v$ which can be solved by $\int \frac{dv}{F(v)-v} = \int \frac{dx}{x}$.

(c) Let $v = y^{1-n}$. We have $\ln(v) = (1 - n)\ln(y)$, $(\ln(v))' = (1 - n)(\ln(y))'$ and $\frac{v'}{v} = (1 - n)\frac{y'}{y}$. Dividing the equation equation $y' + P(x)y = Q(x)y^n$ by y, we have $\frac{y'}{y} + P(x) = Q(x)y^{n-1}$. Use $\frac{v'}{v} = (1 - n)\frac{y'}{y}$ and $v = y^{1-n}$, we have $\frac{y'}{y} = \frac{1}{1-n}\frac{v'}{v}$ and $y^{n-1} = v^{-1}$. So the equation $\frac{y'}{y} + P(x) = Q(x)y^{n-1}$ can be written as $\frac{1}{1-n}\frac{v'}{v} + P(x) = Q(x)v^{-1}$. Multiplying $\frac{1}{1-n}\frac{v'}{v} + P(x) = Q(x)v^{-1}$ by $(1 - n)v$, we obtain
$v' + (1 - n)P(x)v = (1 - n)Q(x)$.

(2) (a) Since $\int \frac{dy}{(y-1)^{\frac{2}{3}}} = \int 6tdt$, we have $y = (t^2 + C)^3 + 1$.

(b) Since $t\frac{dy}{dt} = y(1 + 2t^2)$ and $\int \frac{dy}{y} = \int \frac{1+2t^2}{t}dt$, we have $y(t) = Cte^{t^2}$.

(c) Since $(e^{2t}y)' = e^{2t}(-2t + 1) = -2te^{2t} + e^{2t}$, we have $y = \frac{\int(-2te^{2t}+e^{2t})dt}{e^{2t}} = \frac{e^{2t}-te^{2t}+C}{e^{2t}} = 1 - t + Ce^{-2t}$.

(d) Since $(e^{-3t}y)' = e^{-3t}\cos(2t)$, we have $y = \frac{\int e^{-3t}\cos(2t)dt}{e^{-3t}} = -\frac{3}{13}\cos(2t) + \frac{2}{13}\sin(2t) + Ce^{3t}$.

(e) $y(t) = \frac{1}{5}e^{-2t} + Ce^{3t}$

(f) Note that $\int \frac{4t}{t^2+1}dt = 2\ln(t^2 + 1)$ and $e^{2\ln(t^2+1)} = (t^2 + 1)^2$. $y(t) = \frac{t^4+2t^2+C}{(t^2+1)^2}$

(g) $y(t) = -\frac{t^2}{5} + Ct^{-3}$.

(h) $y(t) = \frac{2}{5}(t + 1)^{\frac{3}{2}} + C(t + 1)^{-1}$

(i) Rewrite the equation $t\frac{dy}{dt} - 6y = 12t^4y^2$ as $\frac{dy}{dt} - \frac{6}{t}y = 12t^3y^2$. Let $v = y^{1-2} = y^{-1}$. We have $\frac{dv}{dt} + \frac{6}{t}v = -12t^3$ and $v(t) = -\frac{6}{5}t^4 + Ct^{-6}$. Thus $y(t) = \frac{1}{v} = \frac{1}{-\frac{6}{5}t^4+Ct^{-6}}$.

(j) Let $v = x + y$. Since $\frac{dv}{dx} = 1 + \frac{dy}{dx}$ and $\frac{dy}{dx} = (x + y)^2 = v^2$, we have $\frac{dv}{dx} = 1 + v^2$. Thus $\int \frac{dv}{1+v^2} = \int dx$ and $\arctan(v) = x + C$. Thus $v = \tan(x + C)$ and $y = v - x = \tan(x + C) - x$.

(k) Let $v = x+y$. Since $\frac{dv}{dx} = 1+\frac{dy}{dx}$ and $\frac{dy}{dx} = \frac{1}{(x+y)^2} = \frac{1}{v^2}$, we have $\frac{dv}{dx} = 1+\frac{1}{v^2} = \frac{v^2+1}{v^2}$. Thus $\int \frac{v^2}{1+v^2}dv = \int dx$, $\int(1 - \frac{1}{1+v^2})dv = \int dx$ and
$v - \arctan(v) = x + C$. Hence $x + y - \arctan(x + y) = x + C$ and
$y = \arctan(x + y) + C$.

(l) Let $v = \frac{y}{x}$, i.e. $y = xv$. Using $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and $\frac{dy}{dx} = \frac{y-2\sqrt{x^2+y^2}}{x} = v - 2\sqrt{1 + v^2}$, we have $v + x\frac{dv}{dx} = v - 2\sqrt{1 + v^2}$. It can be rewritten as $x\frac{dv}{dx} = -2\sqrt{1 + v^2}$ which can be solved by $\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{-2}{x}dx$. Thus $\ln(v + \sqrt{1 + v^2}) = -2\ln(x) + c$

and $v+\sqrt{1+v^2}=Cx^{-2}$. Note that $\int \frac{dv}{\sqrt{1+v^2}}=\ln(v+\sqrt{1+v^2})$ by substituting $v=\tan(\theta)$. This implies that $\sqrt{1+v^2}=Cx^{-2}-v$, $1+v^2=C^2x^{-4}-2Cx^{-2}v+v^2$ and $v=\frac{x^2}{2C}-\frac{Cx^{-2}}{2}$. Hence $y=xv=\frac{x^3}{2C}-\frac{Cx^{-1}}{2}$.

(m) Let $v=\frac{y}{x}$, i.e. $y=xv$. Using $\frac{dy}{dx}=v+x\frac{dv}{dx}$ and $\frac{dy}{dx}=\frac{x-y}{x+y}=\frac{1-v}{1+v}$, we have $v+x\frac{dv}{dx}=\frac{1-v}{1+v}$. It can be rewritten as $x\frac{dv}{dx}=\frac{1-v}{1+v}-v=\frac{1-v-v+v^2}{1+v}=\frac{(v-1)^2}{1+v}$ which can be solved by $\int \frac{v+1}{(v-1)^2}dv=\int dx$. Thus $\int(\frac{1}{v-1}-2\frac{1}{(v-1)^2}dv=\int dx$ $\ln(v-1)-2\frac{1}{v-1}=x+c$. $v+\sqrt{1+v^2}=Cx^{-2}$. Hence $\ln(\frac{y}{x}-1)-2\frac{1}{\frac{y}{x}-1}=x+c$.

(n) The equation $\frac{dy}{dt}=\frac{y^3-6ty}{4y+3t^2-3ty^2}$ can be rewritten as $(4y+3t^2-3ty^2)dy-(y^3-6ty)dt=0$. Let $M(t,y)=4y+3t^2-3ty^2$ and $N(t,y)=-(y^3-6ty)$. Note that $\frac{\partial M}{\partial t}=6t-3y^2$. and $\frac{\partial N}{\partial y}=-3y^2+6t$. Thus $(4y+3t^2-3ty^2)dy-(y^3-6ty)dt=0$ is a exact equation. We have $y^2+3t^2y-ty^3=C$.

(3) It is obvious that $y=0$ and $y=3$ are the only equilibrium solutions. Since $1+t^2+y^2\ge 1>0$, we have $\frac{dy}{dt}=y(3-y)(1+t^2+y^2)>0$ if $0<y<3$. Thus y is increasing if $0<y<3$. By the uniqueness of ODE, we know that $0<y(t)<3$ if $0<y(0)<3$. Now $y(0)=2$, so $0<y(t)<3$ and $y(t)$ is increasing. Thus $\lim_{t\to\infty} y(t)=L$ exists. We have $0<L\le 3$. Suppose $0<L<3$, we have $\lim_{t\to\infty} y'(t)=\lim_{t\to\infty} y(3-y)(1+t^2+y^2)=L(3-L)\lim_{t\to\infty}(1+t^2+y^2)=\infty$. But this will imply that $\lim_{t\to\infty} y(t)=\infty$. Therefore $L=3$ and $\lim_{t\to\infty} y(t)=3$.

(4) (a) Let $f(y)=y^3-3y^2+2y$. We have $f(y)=y^3-3y^2+2y=y(y^2-3y+2)=y(y-1)(y-2)$. Thus $f(y)<0$ when $y\in(-\infty,0)\cup(1,2)$ and $f(y)>0$ when $y\in(0,1)\cup(2,\infty)$. Therefore $\{1\}$ is an asymptotically stable equilibrium point and$\{0,2\}$ are unstable equilibrium points.

(b) Let $f(y)=(y^3-3y^2+2y)(y-3)^2$. We have $f(y)=(y^3-3y^2+2y)(y-3)^2=y(y-1)(y-2)(y-3)^2$. Thus $f(y)<0$ when $y\in(-\infty,0)\cup(1,2)$ and $f(y)>0$ when $y\in(0,1)\cup(2,3)\cup(3,\infty)$. Therefore $\{1\}$ is an asymptotically stable equilibrium point,$\{0,2\}$ are unstable equilibrium points and $\{3\}$ is a semistable equilibrium point.

(5) (a) $y(t)=c_1e^{-2t}+c_2e^{-3t}$.

(b) $y(t)=c_1e^{-2t}+c_2te^{-2t}$.

(c) $y(t)=c_1e^{-2t}\cos(2t)+c_2te^{-2t}\sin(2t)$.

(d) The characteristic equation of $y^{(6)}(t)+64y(t)=0$ is $r^6+64=0$. Note that $-64=64e^{i(\pi+2k\pi)}$ where k is an integer. Solving $r^6+64=0$ is the same as solving $r^6=-64=64e^{i(\pi+2k\pi)}$. Therefore $r=\sqrt[6]{64}e^{i\frac{(\pi+2k\pi)}{6}}=2e^{i\frac{(\pi+2k\pi)}{6}}=2(\cos(\frac{(\pi+2k\pi)}{6})+i\sin(\frac{(\pi+2k\pi)}{6}))$ where $k=0,1,2,\cdots,5$.

Let $r_k=2(\cos(\frac{(\pi+2k\pi)}{6})+i\sin(\frac{(\pi+2k\pi)}{6}))$ where $k=0,1,2,\cdots,5$. Therefore

$r_0 = 2(\cos(\frac{\pi}{6})+i\sin(\frac{\pi}{6})) = \sqrt{3}+i$, $r_1 = 2(\cos(\frac{\pi}{2})+i\sin(\frac{\pi}{2})) = 2i$, $r_2 = 2(\cos(\frac{5\pi}{6})+i\sin(\frac{5\pi}{6})) = -\sqrt{3}+i$, $r_3 = 2(\cos(\frac{7\pi}{6})+i\sin(\frac{7\pi}{6})) = -\sqrt{3}-i$, $r_4 = 2(\cos(\frac{3\pi}{2})+i\sin(\frac{3\pi}{2})) = -2i$ and $r_5 = 2(\cos(\frac{11\pi}{6})+i\sin(\frac{11\pi}{6})) = \sqrt{3}-i$. Note that $r_0 = \overline{r_5}$, $r_1 = \bar{r_4}$ and $r_2 = \bar{r_3}$.

The general solution is $y(t) = c_1 e^{\sqrt{3}t}\cos(t)+c_2 e^{\sqrt{3}t}\sin(t)+c_3\cos(2t)+c_4\sin(2t)+c_5 e^{-\sqrt{3}t}\cos(t)+c_6 e^{-\sqrt{3}t}\sin(t)$. This solution is unstable.

(e) $y(t) = c_1 e^{-2t}\cos(2t) + c_2 e^{-2t}\sin(2t) + c_3 te^{-2t}\cos(2t) + c_4 te^{-2t}\sin(2t) + c_5 e^{2t} + c_6 te^{2t} + c_7 t^2 e^{2t} + c_8 + c_9 t$. This solution is unstable.

(f) Try $y = t^r$. We have $y' = rt^r$, $y'' = r(r-1)t^{r-2}$ and $t^2y''(t) + 2ty'(t) - 2y = r(r-1)t^r + 2rt^r - 2t^r = (r^2+r-2)t^r = 0$ if $r^2+r-2 = (r+2)(r-1) = 0$, ie. $r = 2$ and $r = -1$ So $y(t) = c_1 t^{-2} + c_2 t$.

(g) $t^2y''(t) + 2ty'(t) - 2y = 0$. Try $y = t^r$. We have $y' = rt^r$, $y'' = r(r-1)t^{r-2}$ and $t^2y''(t) + 5ty'(t) + 4y = r(r-1)t^r + 5rt^r + 4t^r = (r^2+4r+4)t^r = 0$ if $r^2+4r+4 = (r+2)^2 = 0$, ie. $r = -2$ and $r = -2$ So $y(t) = c_1 t^{-2} + c_2 t^{-2}\ln(t)$.

(h) $t^2y''(t) + 5ty'(t) + 8y = 0$. Try $y = t^r$. We have $y' = rt^r$, $y'' = r(r-1)t^{r-2}$ and $t^2y''(t) + 5ty'(t) + 8y = r(r-1)t^r + 5rt^r + 4t^r = (r^2+4r+8)t^r = 0$ if $r^2+4r+8 = 0$, ie. $r = -2\pm 2i$. So $t^{-2\pm 2i} = t^{-2}\cos(2\ln(t)) + it^{-2}\sin(2\ln(t))$. So $y(t) = c_1 t^{-2}\cos(2\ln(t)) + c_2 t^{-2}\sin(2\ln(t))$.

(i) $t^3y'''(t) - 3ty'(t) + 3y = 0$. Try $y = t^r$. We have $y' = rt^r$, $y'' = r(r-1)t^{r-2}$, $y''' = r(r-1)(r-2)t^{r-3}$ and $t^3y'''(t) - 3ty'(t) + 3y == r(r-1)(r-2)t^r - 3rt^r + 3t^r = (r^3-3r^2-r+3)t^r = 0$ if $(r^3-3r^2-r+3) = r^2(r-3)-(r-3) = (r^2-1)(r-3) = (r-1)(r+1)(r-3) = 0$ and $r = -1$, $r = 1$ and $r = 3$. So $y(t) = c_1 t^{-1} + c_2 t + c_3 t^3$.

(j) $y''(t) + 5y'(t) + 4y = g(t)$ with $y(0) = 0$ and $y'(0) = 0$ where

$$g(t) = \begin{cases} 0, & 0 \le t < 2, \\ 3(t-2), & 2 \le t < 4, \\ 6, & 4 \le t. \end{cases}$$

$$g(t) = \begin{cases} 0, & 0 \le t < 2, \\ 3(t-2), & 2 \le t < 4, \\ 6, & 4 \le t. \end{cases}$$

We have $g(t) = 3(t-2)u_{2,4}(t) + 6u_4(t) = 3(t-2)(u_2(t) - u_4(t)) + 6u_4(t) = 3(t-2)u_2(t) - (3t-12)u_4(t) = 3(t-2)u_2(t) - 3(t-4)u_4(t)$. Let $h(t-2) = t-2$ and $k(t-4) = t-4$. Then $h(t) = t$ and $k(t) = t$. So $g(t) = 3h(t-2)u_2(t) - 3k(t-4)u_4(t)$ and $L(g(t)) = L(3h(t-2)u_2(t) - 3k(t-4)u_4(t)) = 3e^{-2s}L(h(t)) - 3e^{-4s}L(k(t)) = 3\frac{e^{-2s}}{s^2} - 3\frac{e^{-4s}}{s^2}$.

$L(y''(t) + 5y'(t) + 4y(t)) = L(g(t)) = 3\frac{e^{-2s}}{s^2} - 3\frac{e^{-4s}}{s^2}$

$\Rightarrow (s2 + 5s + 4)Y(s) = 3\frac{e^{-2s}}{s^2} - 3\frac{e^{-4s}}{s^2}$

$\Rightarrow Y(s) = 3\frac{e^{-2s}}{s^2(s^2+5s+4)} - 3\frac{e^{-4s}}{s^2(s^2+4s+4)}$
$\Rightarrow Y(s) = 3\frac{e^{-2s}}{s^2(s+1)(s+4)} - 3\frac{e^{-4s}}{s^2(s+1)(s+4)}$
Using partial fraction, we have $\frac{1}{s^2(s+1)(s+4)} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s+1} + \frac{d}{s+4}$. This implies that $1 = as(s+1)(s+4) + b(s+1)(s+4) + cs^2(s+4) + ds^2(s+1)$. Plugging in $s = 0$, we get $b = \frac{1}{4}$. Plugging in $s = -1$, we get $c = \frac{1}{3}$. Plugging in $s = -4$, we get $d = -\frac{1}{48}$. Hence $1 = as(s+1)(s+4) + \frac{1}{4}(s+1)(s+4) + \frac{1}{3}s^2(s+4) - \frac{1}{48}s^2(s+1)$. Plugging in $s = 1$, we have $1 = 10a + \frac{5}{2} + \frac{5}{3} - \frac{1}{24}$. This gives $a = -\frac{5}{16}$. Now we have $\frac{1}{s^2(s+1)(s+4)} = -\frac{5}{16}\frac{1}{s} + \frac{1}{4}\frac{1}{s^2} + \frac{1}{3}\frac{1}{s+1} - \frac{1}{48}\frac{1}{s+4}$. Let $f(t) = L^{-1}(-\frac{5}{16}\frac{1}{s} + \frac{1}{4}\frac{1}{s^2} + \frac{1}{3}\frac{1}{s+1} - \frac{1}{48}\frac{1}{s+4}) = -\frac{5}{16} + \frac{1}{4}t + \frac{1}{3}e^{-t} + -\frac{1}{48}e^{-4t}$. Then $y(t) = L^{-1}(3\frac{e^{-2s}}{s^2(s+1)(s+4)} - 3\frac{e^{-4s}}{s^2(s+1)(s+4)}) = 3u_2(t)f(t-2) - 3u_4(t)f(t-4)$.

(k) $y''(t) + 4y'(t) + 5y = g(t)$ with $y(0) = 0$ and $y'(0) = 0$ where

$$g(t) = \begin{cases} 0, & 0 \le t < 2, \\ 1, & 2 \le t < 4, \\ 0, & 4 \le t. \end{cases}$$

We have $g(t) = u_{2,4}(t) = u_2(t) - u_4(t)$ and $L(g(t)) = e^{-2s} - e^{-4s}$. Taking the Laplace transform, we get $L(y''(t)+5y'(t)+5y) = L(g(t))$ and $(s^2+4s+5)Y(s) = e^{-2s} - e^{-4s}$.
$\Rightarrow Y(s) = \frac{e^{-2s}}{(s^2+4s+5)} - \frac{e^{-4s}}{(s^2+4s+5)}$.
Note that $\frac{1}{(s^2+4s+5)} = \frac{1}{(s+2)^2+1}$ and $f(t) = L^{-1}(\frac{1}{(s+2)^2+1}) = e^{-2t}\sin(t)$.
Then $y(t) = L^{-1}(\frac{e^{-2s}}{(s^2+4s+5)} - \frac{e^{-4s}}{(s^2+4s+5)}) = u_2(t)f(t-2) - u_4(t)f(t-4)$.

(l) $y''(t) + 5y'(t) + 4y(t) = \delta(t-2)$, with $y(0) = 0$ and $y'(0) = 0$.
Taking the Laplace transform $L(y''(t) + 5y'(t) + 4y(t)) = L(\delta(t-2))$, we have $(s^2 + 5s + 4)Y(s) = e^{-2s}$.
$\Rightarrow Y(s) = \frac{e^{-2s}}{(s^2+5s+4)} = \frac{e^{-2s}}{((s+1)(s+4))} = e^{-2s}(\frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s+4})$
$= \frac{1}{3}\frac{e^{-2s}}{s+1} - \frac{1}{3}\frac{e^{-2s}}{s+4}$.
Let $f(t) = L^{-1}(\frac{1}{s+1}) = e^{-t}$ and $g(t) = L^{-1}(\frac{1}{s+4}) = e^{-4t}$
We have $y(t) = L^{-1}(\frac{1}{3}\frac{e^{-2s}}{s+1} - \frac{1}{3}\frac{e^{-2s}}{s+4}) = \frac{1}{3}u_2(t)f(t-2) - \frac{1}{3}u_2(t)g(t-2) = \frac{1}{3}u_2(t)e^{-(t-2)}) - \frac{1}{3}u_2(t)e^{-4(t-2)}$.

(m) $y''(t) + 4y'(t) + 5y(t) = \delta(t-2)$, with $y(0) = 1$ and $y'(0) = 1$.
Taking the Laplace transform $L(y''(t) + 4y'(t) + 5y(t)) = L(\delta(t-2))$, we have $(s^2 + 4s + 5)Y(s) - s - 5 = e^{-2s}$.
$\Rightarrow Y(s) = \frac{e^{-2s}(s+5)}{(s^2+4s+5)} + \frac{e^{-2s}}{(s^2+4s+5)} = \frac{e^{-2s}(s+5)}{(s+2)^2+1} + \frac{e^{-2s}}{(s+2)^2+1}$
$= \frac{e^{-2s}(s+2)}{(s+2)^2+1} + 4\frac{e^{-2s}}{(s+2)^2+1}$.
Let $f(t) = L^{-1}(\frac{(s+2)}{(s+2)^2+1}) = e^{-2t}\cos(t)$ and $g(t) = L^{-1}(\frac{1}{(s+2)^2+1}) = e^{-2t}\sin(t)$

We have $y(t) = L^{-1}(\frac{e^{-2s}(s+2)}{(s+2)^2+1} + 4\frac{e^{-2s}}{(s+2)^2+1}) = u_2(t)f(t-2) + 4u_2(t)g(t-2) = u_2(t)e^{-2(t-2)}\cos(t-2) + 4u_2(t)e^{-2(t-2)}\sin(t-2)$.

(n) $y''(t) - 4y'(t) + 4y(t) = \delta(t-2)$, with $y(0) = 0$ and $y'(0) = 0$.
Taking the Laplace transform $L(y''(t) + 4y'(t) + 4y(t)) = L(\delta(t-2))$, we have $(s^2+4s+4)Y(s) = e^{-2s}$.
$\Rightarrow Y(s) = \frac{e^{-2s}}{(s^2+4s+4)} = \frac{e^{-2s}}{(s+2)^2}$.
Let $f(t) = L^{-1}(\frac{1}{(s+2)^2}) = te^{-2t}$.
We have $y(t) = L^{-1}(\frac{e^{-2s}}{(s+2)^2}) = u_2(t)f(t-2) = u_2(t)(t-2)e^{-2(t-2)}$.

(6) Note that the equation in this problem should be
$ty''(t) + (1-2t)y'(t) + (t-1)y(t) = te^t$.

First, we rewrite the equation as $y''(t) + \frac{(1-2t)}{t}y'(t) + \frac{(t-1)}{t}y(t) = e^t$.
First, we find the other solution of $y''(t) + \frac{(1-2t)}{t}y'(t) + \frac{(t-1)}{t}y(t) = 0$.
Given that $y_1(t) = e^t$ is a solution of $y''(t) + \frac{(1-2t)}{t}y'(t) + \frac{(t-1)}{t}y(t) = 0$
Let $p(t) = \frac{(1-2t)}{t}$. Let y_2 be another solution of $y''(t)+\frac{(1-2t)}{t}y'(t)+\frac{(t-1)}{t}y(t) = 0$. We have $(\frac{y_2}{y_1})' = \frac{y_1y_2' - y_1'y_2}{y_1^2} = \frac{W(t)}{y_1^2} = \frac{Ce^{-\int p(t)dt}}{(e^t)^2} = \frac{Ce^{\int \frac{-1+2t}{t}dt}}{e^{2t}} = \frac{Ce^{\int(-\frac{1}{t}+2)dt}}{e^{2t}} = \frac{Ce^{(2t-\ln(t))}}{e^{2t}} = \frac{Ce^{2t}e^{-\ln t}}{e^{2t}} = \frac{Ce^{2t}}{te^{2t}} = C\frac{1}{t}$. So $\frac{y_2}{y_1} = \int C\frac{1}{t}dt = C\ln t + D$ and $y_2 = y_1(C\ln t + D) = e^t(C\ln t + D) = Ce^t\ln t + De^t$. So the general solution is $y = Ce^t\ln t + De^t$. We may choose the second independent solution to be $y_2 = e^t\ln t$.

Now we use variation of parameter to find the general solution. Now $y_1 = e^t$, $y_2 = e^t\ln t$, $y_1' = e^t$, $y_2' = e^t\ln t + \frac{e^t}{t}$ and $W(y_1,y_2)(t) = y_1y_2' - y_2y_1' = e^t\cdot(e^t\ln t + \frac{e^t}{t}) - e^t\ln t\cdot e^t = \frac{e^{2t}}{t}$. Recall that $g(t) = e^t$.
$\int \frac{y_2g(t)}{W(y_1,y_2)(t)}dt = \int \frac{e^t\ln t\cdot e^t}{\frac{e^{2t}}{t}}dt = \int(t\ln t)dt = \frac{t^2}{2}\ln(t) - \frac{t^2}{4} + c$.
$\int \frac{y_1g(t)}{W(y_1,y_2)(t)}dt = \int \frac{e^t\cdot e^t}{\frac{e^{2t}}{t}}dt = \int(t)dt = \frac{t^2}{2} + d$.
Thus $y(t) = -e^t\cdot(\frac{t^2}{2}\ln(t) - \frac{t^2}{4} + c) + e^t\ln t(\frac{t^2}{2} + d) = \frac{t^2}{4}e^t + ce^{-t} + de^t\ln t$.

(7) (a) We should try $y_p(t) = at^2e^{-2t} + be^{3t}$. We have $y_p'(t) = 2ate^{-2t} - 2at^2e^{-2t} + 3be^{3t}$, $y_p''(t) = 2ae^{-2t} - 8ate^{-2t} + 4at^2e^{-2t} + 9be^{3t}$ and
$y_p''(t) + 4y_p'(t) + 4y_p(t) = 2ae^{-2t} + 25be^{3t}$.
Hence $y_p''(t) + 4y_p'(t) + 4y_p(t) = e^{-2t} + e^{3t}$ if $2a = 1$ and $25b = 1$. We have $a = \frac{1}{2}$, $b = \frac{1}{25}$ and $y_p(t) = \frac{1}{2}t^2e^{-2t} + \frac{1}{25}e^{3t}$.
The general solution is $y(t) = \frac{1}{2}t^2e^{-2t} + \frac{1}{25}e^{3t} + c_1e^{-2t} + c_2te^{-2t}$.

(b) We will use the variation of parameter formula. We have $y_1(t) = e^{-2t}$, $y_2(t) = te^{-2t}$.
$W(y_1,y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = e^{-2t}\cdot(e^{-2t} - 2te^{-2t}) - te^{-2t}\cdot(-2e^{-2t}) = e^{-4t}$,
$\int \frac{y_2g(t)}{W(y_1,y_2)(t)}dt = \int \frac{te^{-2t}e^{-2t}\ln t}{e^{-4t}}dt$

$= \int t\ln t dt = \frac{1}{2}t^2\ln t - \frac{t^2}{4} + c$, $\int \frac{y_1 g(t)}{W(y_1,y_2)(t)}dt = \int \frac{e^{-2t}e^{-2t}\ln t}{e^{-4t}}dt = \int \ln t dt = t\ln t - t + d$

Thus $y(t) = -e^{-2t}\cdot(\frac{1}{2}t^2\ln t - \frac{t^2}{4} + c) + te^{-2t}(t\ln t - t + d)$.

(c) We should try $y_p(t) = at\sin(3t) + bt\cos(3t) + c\cos(2t) + d\sin(2t)$. We have $y_p'(t) = a\sin(3t) + 3at\cos(3t) + b\cos(3t) - 3bt\sin(3t) - 2c\sin(2t) + 2d\cos(2t)$, $y_p''(t) = 6a\cos(3t) - 9at\sin(3t) - 6b\sin(3t) - 9bt\cos(3t) - 4c\cos(2t) - 4d\sin(2t)$ and $y_p''(t) + 9y_p(t) = 6a\cos(3t) - 6b\sin(3t) + 5c\cos(2t) + 5d\sin(2t)$. Hence $y_p''(t) + 9y_p(t) = \sin(3t) + \cos(2t)$ if $6a = 0$, $-6b = 1$, $5c = 1$ and $5d = 0$. We have $a = 0$, $b = -\frac{1}{6}$, $c = \frac{1}{5}$, $d = 0$ and $y_p(t) = -\frac{1}{6}t\cos(3t) + \frac{1}{5}\cos(2t)$. The general solution is $y(t) = -\frac{1}{6}t\cos(3t) + \frac{1}{5}\cos(2t) + c_1\cos(3t) + c_2 t\sin(3t)$.

(d) We will use the variation of parameter formula. We have $y_1(t) = \sin(2t)$, $y_2(t) = \cos(2t)$,

$W(y_1,y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = \sin(2t)\cdot(-2\sin(2t)) - \cos(2t)\cdot(2\cos(2t)) = -2$,

$\int \frac{y_2 g(t)}{W(y_1,y_2)(t)}dt = \int \frac{\cos(2t)\sec^2(2t)}{-2}dt = \int \frac{\cos(2t)}{-2\cos^2(2t)}dt = \int \frac{-1}{2\cos(2t)}dt = -\int \frac{1}{2}\sec(2t)dt = -\frac{\ln|\sec(2t)+\tan(2t)|}{4} + c$ and

$\int \frac{y_1 g(t)}{W(y_1,y_2)(t)}dt = \int \frac{\sin(2t)\sec^2(2t)}{-2}dt = \int \frac{\sin(2t)}{-2\cos^2(2t)}dt$

$= -\frac{1}{4\cos(2t)} + d = -\frac{1}{4}\sec(2t) + d$. We have used substitution $u = \cos(2t)$ and $du = -2\sin(2t)dt$.

Thus $y(t) = -\sin(2t)\cdot(-\frac{\ln|\sec(2t)+\tan(2t)|}{4} + c) + \cos(2t)(-\frac{1}{4}\sec(2t) + d)$

$= \sin(2t)\frac{\ln|\sec(2t)+\tan(2t)|}{4} + \frac{1}{4} + C\sin(2t) + D\cos(2t)$.

(e) We should try $y_p(t) = ate^{-2t} + be^{3t} + c\sin(t) + d\cos(t)$. We have

$y_p'(t) = ae^{-2t} - 2ate^{-2t} + 3be^{3t} - c\sin(t) + d\cos(t)$,

$y_p''(t) = -4ae^{-2t} + 4ate^{-2t} + 9be^{3t} - c\cos(t) - d\sin(t)$ and

$y_p''(t) + 5y_p'(t) + 6y_p(t) = 2ae^{-2t} + 30be^{3t} + 5c\cos(t) + 5d\sin(t) - 5c\sin(t) + 5d\cos(t)$.

Hence $y_p''(t) + 5y_p'(t) + 6y_p(t) = 4e^{-2t} + e^{3t} + \sin(t)$ if $a = 1$, $30b = 1$, $5c + 5d = 0$ and $5d - 5c = 1$ We have $a = 1$, $b = \frac{1}{30}$,$c = -\frac{1}{10}$, $d = \frac{1}{10}$ and

$y_p(t) = te^{-2t} + \frac{1}{30}e^{3t} - \frac{1}{10}\sin(t) + \frac{1}{10}\cos(t)$.

The general solution is $y(t) = te^{-2t} + \frac{1}{30}e^{3t} - \frac{1}{10}\sin(t) + \frac{1}{10}\cos(t) + c_1e^{-2t} + c_2te^{-3t}$.

(f) First we solve the homogeneous equation $t^2y''(t) - 4ty'(t) + 6y(t) = 0$. Let $y(t) = t^r$, we have $y'(t) = rt^{r-1}$ and $y''(t) = r(r-1)t^{r-2}$.

Thus $t^2y''(t) - 4ty'(t) + 6y(t) = (r(r-1) - 4r + 6)t^r = (r^2 - 5r + 6)t^r$.

So $y = t^r$ is a solution of $t^2y''(t) - 4ty'(t) + 6y(t) = 0$ if $r^2 - 5r + 6 = (r-3)(r-2) = 0$. We have $y_1(t) = t^3$ and $y_2(t) = t^2$.

Thus $W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = t^3 \cdot 2t - t^2(3t^2) = -t^4$.
The equation $t^2y''(t) - 4ty'(t) + 6y = t^3 + 1$ can be rewritten as $y''(t) - \frac{4}{t}y'(t) + \frac{6}{t^2}y(t) = t + \frac{1}{t^2} = t + t^{-2}$.
$\int \frac{y_2 g(t)}{W(y_1,y_2)(t)}dt = \int \frac{t^2(t+t^{-2})}{-t^4}dt = -\ln|t| + \frac{1}{3}t^{-3} + c$ and
$\int \frac{y_1 g(t)}{W(y_1,y_2)(t)}dt = \int \frac{t^3(t+t^{-2})}{-t^4}dt = -t + \frac{1}{2}t^{-2} + d$.
The general solution is
$y(t) = -t^3(-\ln|t| + \frac{1}{3}t^{-3} + c) + t^2(-t + \frac{1}{2}t^{-2} + d)$.

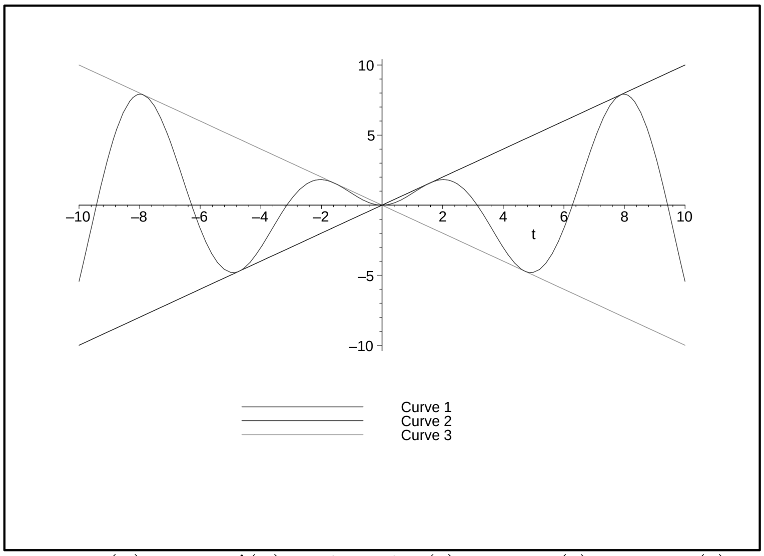
(8) (a) The equation $y''(t) - 5y'(t) + 6y(t) = te^{2t} + e^{3t} + e^{-2t}$ can be rewritten as $(D^2 - 5D + 6)y(t) = te^{2t} + e^{3t} + e^{-2t}$. We divide this into three equations $(D^2 - 5D + 6)y(t) = te^{2t}$, $(D^2 - 5D + 6)y(t) = e^{3t}$ and $(D^2 - 5D + 6)y(t) = e^{-2t}$. Using $(D-2)^2(te^{2t}) = 0$, $(D-3)e^{3t} = 0$ and $(D+2)e^{-2t}$, we get $(D-2)^2(D^2-5D+6)y(t) = (D-2)^2(D-2)(D-3)y(t) = 0$, $(D-3)(D^2-5D+6)y(t) = (D-3)(D-2)(D-3)y(t) = 0$ and $(D+2)(D^2-5D+6)y(t) = (D+2)(D-2)(D-3)y(t) = 0$. Thus the particular solution for $(D^2 - 5D + 6)y(t) = te^{2t} + e^{3t} + e^{-2t}$ is
$y_p(t) = c_1t^2e^{2t} + c_2te^{2t} + c_3te^{3t} + c_4e^{-2t}$.

(b) The equation $y''(t) + 4y = t\sin(2t) - 3\cos(t)$ can be rewritten as $(D^2+4)y(t) = t\sin(2t) - 3\cos(t)$. We divide this into two equations $(D^2+4)y(t) = t\sin(2t)$ and $(D^2+4)y(t) = -3\cos(t)$. Using $(D^2+4)^2(t\sin(2t)) = 0$ and $(D^2+1)\cos(t) = 0$, we get $(D^2+4)^2(D^2+4)y(t) = 0$ and $(D^2+1)(D^2+4)y(t) = 0$. Thus the particular solution for $(D^2+4)y(t) = t\sin(2t) - 3\cos(t)$ is
$y_p(t) = c_1t^2\cos(2t) + c_2t^2\sin(2t) + c_3t\cos(2t) + c_4t\sin(2t) + c_5\cos(t) + c_6\sin(t)$.

(c) The equation $y''(t) - 4y'(t) + 5y(t) = e^{2t}\sin(t) + e^{3t}\sin(t)$ can be rewritten as $(D^2-4D+5)y(t) = e^{2t}\sin(t) + e^{3t}\sin(t)$. We divide this into two equations $(D^2-4D+5)y(t) = e^{2t}\sin(t)$ and $(D^2-4D+5)y(t) = e^{3t}\sin(t)$. Note that $D^2-4D+5 = (D-2)^2+1$. Using $((D-2)^2+1)(e^{2t}\sin(t)) = 0$ and $((D-3)^2+1)e^{3t}\sin(t = 0$, we get $((D-2)^2+1)((D-2)^2+1)y(t) = 0$ and $((D-3)^2+1)((D-2)^2+1)y(t) = 0$. Thus the particular solution for $(D^2-4D+5)y(t) = e^{2t}\sin(t) + e^{3t}\sin(t)$ is
$y_p(t) = c_1te^{2t}\cos(t) + c_2te^{2t}\sin(t) + c_3e^{3t}\cos(t) + c_4e^{3t}\sin(t)$.

(9) (a) The equation $y''(t) + y(t) = 2\cos(t)$ can be rewritten as $(D^2+1)y(t) = 2\cos(t)$. Since $(D^2+1)\cos(t) = 0$, we have $(D^2+1)(D^2+1)y(t) = (D^2+1)^2\cos(t) = 0$. Thus the particular solution is of the form $y_p(t) = ct\cos(t) + dt\sin(t)$. We have
$y_p'(t) = c\cos(t) - ct\sin(t) + d\sin(t) + dt\cos(t)$,
$y_p''(t) = -c\sin(t) - c\sin(t) - ct\cos(t) + d\cos(t) + d\cos(t) - dt\sin(t) = -2c\sin(t) - ct\cos(t) + 2d\cos(t) - dt\sin(t)$
and $y_p''(t) + y_p(t) = -2c\sin(t) + 2d\cos(t)$.
Hence $y_p''(t) + y_p(t) = 2\cos(t)$ if $c = 0$ and $d = 1$. Thus $y_p(t) = t\sin(t)$ and
$y(t) = t\sin(t) + c_1\cos(t) + c_2\sin(t)$.

(b) Using $y(0) = 0$, $y'(0) = 0$ and $y(t) = t\sin(t) + c_1\cos(t) + c_2\sin(t)$, we have $c_1 = 0$, $c_2 = 0$ and $y(t) = t\sin(t)$. The solution oscillate between $-t$ and t.



(c) Using $y(0) = a$, $y'(0) = b$ and $y(t) = t\sin(t) + c_1\cos(t) + c_2\sin(t)$, we have $c_1 = a$, $c_2 = b$ and $y(t) = t\sin(t) + a\cos(t) + b\sin(t)$. The solution will not be bounded for any a and b.

10 (a) $y''(t) + 4y'(t) + 5y(t) = e^{2t}\cos(t)$, with $y(0) = 0$ and $y'(0) = 0$.
Taking the Laplace transform of the equation, we have
$L(y''(t) + 4y'(t) + 5y(t)) = L(e^{2t}\cos(t))$
$\Rightarrow (s^2 + 4s + 5)Y(s) = \frac{(s-2)}{(s-2)^2+1}$
$\Rightarrow Y(s) = \frac{(s-2)}{((s-2)^2+1)}\frac{1}{(s^2+4s+5)}$. Let $f(t) = L^{-1}(\frac{(s-2)}{(s-2)^2+1}) = e^{2t}\cos(t)$ and $g(t) = L^{-1}(\frac{1}{s^2+4s+5}) = L^{-1}(\frac{1}{(s+2)^2+1}) = e^{-2t}\sin(t)$. So $y(t) = \int_0^t f(t-\tau)g(\tau)d\tau$.

(b) $y''(t) - 2y'(t) + y(t) = te^t$, with $y(0) = 0$ and $y'(0) = 0$.
Taking the Laplace transform of the equation, we have
$L(y''(t) - 2y'(t) + y(t)) = L(te^t)$
$\Rightarrow (s^2 - 2s + 1)Y(s) = \frac{1}{(s-1)^2}$
$\Rightarrow Y(s) = \frac{1}{(s-1)^2}\frac{1}{(s^2-2s+1)} = \frac{1}{(s-1)^2} \cdot \frac{1}{(s-1)^2}$. Let $f(t) = L^{-1}(\frac{1}{(s-1)^2}) = te^t$
So $y(t) = \int_0^t f(t-\tau)f(\tau)d\tau$.

(c) $y''(t) - 3y'(t) + 2y(t) = te^t + te^{2t}$, with $y(0) = 0$ and $y'(0) = 0$.
Taking the Laplace transform of the equation, we have
$L(y''(t) - 3y'(t) + 2y(t)) = L(te^t + te^{2t})$
$\Rightarrow (s^2 - 3s + 2)Y(s) = \frac{1}{(s-1)^2} + \frac{1}{(s-2)^2}$
$\Rightarrow Y(s) = (\frac{1}{(s-1)^2} + \frac{1}{(s-2)^2})\frac{1}{(s^2-2s+1)} = (\frac{1}{(s-1)^2} + \frac{1}{(s-2)^2}) \cdot \frac{1}{(s-1)^2}$. Let $f(t) = L^{-1}(\frac{1}{(s-1)^2} + \frac{1}{(s-2)^2}) = te^t + te^{2t}$ and $g(t) = L^{-1}(\frac{1}{(s-1)^2}) = te^t$.

So $y(t) = \int_0^t f(t-\tau)g(\tau)d\tau$.

(11) (a)

$$\text{Let} \quad A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

$$det(A - \lambda I) = det \begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix} = (a-\lambda)^2 - b^2 = (a-\lambda-b)(a-\lambda+b).$$

Therefore the characteristic equation is $(a-\lambda-b)(a-\lambda+b) = 0$. Hence the eigenvalues of A are $\lambda = a+b$ and $\lambda = a-b$.

To find the eigenvector corresponding to $\lambda = a+b$, we must solve $(A-\lambda I)v = 0$. Substituting A and $\lambda = a+b$ gives

$$\begin{pmatrix} a-(a+b) & b \\ b & a-(a+b) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -b & b \\ b & -b \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This matrix equation is equivalent to the single equation $bv_1 - bv_2 = 0$. Therefore $v_2 = v_1$ and

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

To find the eigenvector corresponding to $\lambda = a-b$, we must solve $(A-\lambda I)v = 0$. Substituting A and $\lambda = a-b$ gives

$$\begin{pmatrix} a-(a-b) & b \\ b & a-(a-b) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} b & b \\ b & b \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This matrix equation is equivalent to the single equation $bv_1 + bv_2 = 0$. Therefore $v_2 = -v_1$ and

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus the general solution is

$$x(t) = c_1 e^{(a+b)t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{(a-b)t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} c_1 e^{(a+b)t} + c_2 e^{(a-b)t} \\ c_1 e^{(a+b)t} - c_2 e^{(a-b)t} \end{pmatrix}.$$

(b)

$$\text{Let} \quad A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

$$det(A - \lambda I) = det \begin{pmatrix} a-\lambda & -b \\ b & a-\lambda \end{pmatrix} = (a-\lambda)^2 + b^2.$$

Therefore the characteristic equation is $(a-\lambda)^2 + b^2 = 0$. Hence the eigenvalues of A are $\lambda = a + ib$ and $\lambda = a - ib$.

To find the eigenvector corresponding to $\lambda = a+ib$, we must solve $(A-\lambda I)v = 0$. Substituting A and $\lambda = a + ib$ gives

$$\begin{pmatrix} a-(a+ib) & b \\ b & a-(a+ib) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -ib & b \\ b & -ib \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This matrix equation is equivalent to the single equation $-ibv_1 + bv_2 = 0$. Therefore $v_2 = iv_1$ and

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ iv_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

The expression

$$e^{(a+bi)t} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

can be simplified as

$$(e^{at}\cos(bt) + ie^{at}\sin(bt)) \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

$$= \begin{pmatrix} e^{at}\cos(bt) \\ -e^{at}\sin(bt) \end{pmatrix} + i \begin{pmatrix} e^{at}\sin(bt) \\ e^{at}\cos(bt) \end{pmatrix}$$

Thus the general solution is

$$x(t) = c_1 \begin{pmatrix} e^{at}\cos(bt) \\ -e^{at}\sin(bt) \end{pmatrix} + c_2 \begin{pmatrix} e^{at}\sin(bt) \\ e^{at}\cos(bt) \end{pmatrix} = \begin{pmatrix} c_1e^{at}\cos(bt) + c_2e^{at}\sin(bt) \\ -c_1e^{at}\sin(bt) + c_2e^{at}\cos(bt) \end{pmatrix}.$$

(c)

$$\text{Let} \quad A = \begin{pmatrix} -5 & 2 \\ -4 & 1 \end{pmatrix}.$$

$$det(A - \lambda I) = det \begin{pmatrix} -5-\lambda & 2 \\ -4 & 1-\lambda \end{pmatrix} = (\lambda+1)(\lambda+3).$$

Hence the eigenvalues of A are $\lambda = -1$ and $\lambda = -3$.
To find the eigenvector corresponding to $\lambda = -1$, we must solve $(A - \lambda I)v = 0$. Substituting A and $\lambda = -1$ gives

$$\begin{pmatrix} -4 & 2 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This matrix equation is equivalent to the single equation $-4v_1 + 2v_2 = 0$. Therefore $v_2 = 2v_1$ and

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

To find the eigenvector corresponding to $\lambda = -3$, we must solve $(A - \lambda I)v = 0$. Substituting A and $\lambda = -3$ gives

$$\begin{pmatrix} -2 & 2 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This matrix equation is equivalent to the single equation $-2v_1 + 2v_2 = 0$. Therefore $v_2 = v_1$ and

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus the general solution is

$$x(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} + c_2 e^{-3t} \\ 2c_1 e^{-t} + c_2 e^{-3t} \end{pmatrix}.$$

(d)

$$\text{Let} \quad A = \begin{pmatrix} -2 & -1 \\ 2 & -4 \end{pmatrix}.$$

$$det(A - \lambda I) = det\begin{pmatrix} -2-\lambda & -1 \\ 2 & -4-\lambda \end{pmatrix} = \lambda^2 + 6\lambda + 10.$$

Hence the eigenvalues of A are $\lambda = -3+i$ and $\lambda = -3-i$.
To find the eigenvector corresponding to $\lambda = -3+i$, we must solve $(A-\lambda I)v = 0$. Substituting A and $\lambda = -3+i$ gives

$$\begin{pmatrix} 1-i & -1 \\ 2 & -1-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This matrix equation is equivalent to the single equation $(1-i)v_1 - v_2 = 0$. Therefore $v_2 = (1-i)v_1$ and

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ (1-i)v_1 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ 1-i \end{pmatrix}.$$

The expression

$$e^{(-3+i)t}\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

can be simplified as

$$(e^{-3t}\cos(t) + ie^{-3t}\sin(t))\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]$$

$$= \begin{pmatrix} e^{-3t}\cos(t) \\ e^{-3t}\cos(t) + e^{-3t}\sin(t) \end{pmatrix} + i\begin{pmatrix} e^{-3t}\sin(t) \\ e^{-3t}\sin(t) - e^{-3t}\cos(t) \end{pmatrix}$$

Thus the general solution is

$$x(t) = c_1\begin{pmatrix} e^{-3t}\cos(t) \\ e^{-3t}\cos(t) + e^{-3t}\sin(t) \end{pmatrix} + c_2\begin{pmatrix} e^{-3t}\sin(t) \\ e^{-3t}\sin(t) - e^{-3t}\cos(t) \end{pmatrix}$$

$$= \begin{pmatrix} c_1e^{-3t}\cos(t) + c_2e^{-3t}\sin(t) \\ (c_1-c_2)e^{-3t}\cos(t) + (c_1+c_2)e^{-3t}\sin(t) \end{pmatrix}.$$

(e)

$$\text{Let} \quad A = \begin{pmatrix} -5 & 3 \\ -3 & 1 \end{pmatrix}.$$

$$det(A - \lambda I) = det \begin{pmatrix} -5-\lambda & 3 \\ -3 & 1-\lambda \end{pmatrix} = (\lambda+2)^2.$$

Hence the eigenvalues of A are $\lambda = -2$.
To find the eigenvector corresponding to $\lambda = -2$, we must solve $(A - \lambda I)v = 0$. Substituting A and $\lambda = -2$ gives

$$\begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This matrix equation is equivalent to the single equation $-3v_1 + 3v_2 = 0$. Therefore $v_2 = v_1$ and

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The matrix only has one independent eigenvector. We need to find w such that w solves $(A - \lambda I)w = v$ where

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This yields

$$\begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This matrix equation is equivalent to the single equation $-3w_1 + 3w_2 = 1$. Therefore $w_2 = w_1 + \frac{1}{3}$ and

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_1 + \frac{1}{3} \end{pmatrix}$$

We may choose $w_1 = 0$ to get

$$w = \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix}$$

Thus the general solution is

$$x(t) = c_1 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \left[te^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-2t} \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix} \right] = \begin{pmatrix} c_1 e^{-2t} + c_2 te^{-2t} \\ (c_1 + \frac{1}{3}c_2)e^{-2t} + c_2 te^{-2t} \end{pmatrix}.$$

(f)

$$\text{Let} \quad A = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}.$$

$$det(A - \lambda I) = det\begin{pmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{pmatrix} = (\lambda - 3)^2.$$

Hence the eigenvalues of A are $\lambda = -2$.
To find the eigenvector corresponding to $\lambda = 3$, we must solve $(A - \lambda I)v = 0$. Substituting A and $\lambda = 3$ gives

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This matrix equation is equivalent to the single equation $v_1 - v_2 = 0$. Therefore $v_2 = v_1$ and

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The matrix only has one independent eigenvector. We need to find w such that w solves $(A - \lambda I)w = v$ where

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This yields

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This matrix equation is equivalent to the single equation $w_1 - w_2 = 1$. Therefore $w_2 = w_1 - 1$ and

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_1 - 1 \end{pmatrix}$$

We may choose $w_1 = 0$ to get

$$w = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Thus the general solution is

$$x(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\left[te^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{3t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] = \begin{pmatrix} c_1 e^{3t} + c_2 te^{3t} \\ (c_1 - c_2)e^{3t} + c_2 te^{3t} \end{pmatrix}.$$

(12) (a) From (14d), we know that the general solution is

$$x(t) = \begin{pmatrix} c_1e^{-3t}\cos(t) + c_2e^{-3t}\sin(t) \\ (c_1 - c_2)e^{-3t}\cos(t) + (c_1 + c_2)e^{-3t}\sin(t) \end{pmatrix}.$$

Using $x_1(0) = 1$ and $x_1(0) = -1$, we have $c_1 = 1$ and $c_1 - c_2 = -1$. Thus $c_1 = 1$, $c_2 = 2$ and

$$x(t) = \begin{pmatrix} e^{-3t}\cos(t) + 2e^{-3t}\sin(t) \\ -e^{-3t}\cos(t) + 3e^{-3t}\sin(t) \end{pmatrix}.$$

(b) From (14e)the general solution is

$$x(t) = \begin{pmatrix} c_1e^{-2t} + c_2te^{-2t} \\ (c_1 + \frac{1}{3}c_2)e^{-2t} + c_2te^{-2t} \end{pmatrix}.$$

Using $x_1(0) = 1$ and $x_1(0) = -1$, we have $c_1 = 1$ and $c_1 + \frac{1}{3}c_2 = -1$. Thus $c_1 = 1$, $c_2 = -6$ and

$$x(t) = \begin{pmatrix} e^{-2t} - 6te^{-2t} \\ -e^{-2t} - 6te^{-2t} \end{pmatrix}.$$

(13) (a) (12c) The eigenvalues of A are $\lambda = -3$ and $\lambda = -1$. Since $\lim_{t\to\infty} e^{-3t} = 0$ and $\lim_{t\to\infty} e^{-t} = 0$, we conclude that this linear system is asymptotically stable.

(12d) The eigenvalues of A are $\lambda = -3+i$ and $\lambda = -3-i$. Since $\lim_{t\to\infty} e^{-3t}\cos(t) = 0$ and $\lim_{t\to\infty} e^{-3t}\sin(t) = 0$, we conclude that this linear system is asymptotically stable.

(12e) The eigenvalues of A are $\lambda = -2$ and A has only one eigenvector. Since $\lim_{t\to\infty} e^{-2t} = 0$ and $\lim_{t\to\infty} te^{-2t} = 0$, we conclude that this linear system is asymptotically stable.

(12f) The eigenvalues of A are $\lambda = 3$ and A has only one eigenvector. Since $\lim_{t\to\infty} e^{3t} = \infty$, we conclude that this linear system is unstable.

(b) Let

$$A = \begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix}.$$

Then $det(A - \lambda I) = \lambda^2 - 4\lambda + 3$. Therefore the characteristic equation is $(\lambda - 3)(\lambda - 1) = 0$. Hence the eigenvalues of A are $\lambda = 3$ and $\lambda = 1$. Since $\lim_{t\to\infty} e^{3t} = \infty$ and $\lim_{t\to\infty} e^{t} = \infty$, we conclude that this linear system is unstable.

(c) Let

$$A = \begin{pmatrix} 2 & 1 \\ -2 & 4 \end{pmatrix}.$$

Then $det(A-\lambda I) = \lambda^2 - 6\lambda + 10$. Therefore the characteristic equation is $(\lambda-3)^2+1=0$. Hence the eigenvalues of A are $\lambda = 3+i$ and $\lambda = 3-i$. Since $e^{3t}\cos(t)$ and $e^{3t}\sin(t)$ oscillate between $-\infty$ and ∞ , we conclude that this linear system is unstable.

(d) Let

$$A = \begin{pmatrix} 2 & 4 \\ -2 & 2 \end{pmatrix}.$$

Then $det(A-\lambda I) = \lambda^2+4$. Therefore the characteristic equation is $\lambda^2+4=0$. Hence the eigenvalues of A are $\lambda = 2i$ and $\lambda = -2i$. Since $\cos(2t)$ and $\sin(2t)$ are bounded , we conclude that this linear system is stable.

(14) (a) Since $\frac{d}{dt}(\frac{(x'(t))^2}{2} + \frac{x^4(t)}{4}) = x'(t)\cdot x''(t) + x^3(t)\cdot x'(t)$
$= (x''(t)+x^3(t))\cdot x'(t) = 0$
, we have $\frac{(x'(t))^2}{2} + \frac{x^4(t)}{4} = \frac{(x'(0))^2}{2} + \frac{x^4(0)}{4} = \frac{a^2}{2} + \frac{b^4}{4}$. We have used the fact that $x(t)$ satisfies $x''(t)+x^3(t)=0$, $x'(0)=a$ and $x(0)=b$.

(b) From (a), we have $\frac{(x'(t))^2}{2} + \frac{x^4(t)}{4} = \frac{a^2}{2} + \frac{b^4}{4}$.
Therefore $\frac{(x'(t))^2}{2} \le \frac{a^2}{2} + \frac{b^4}{4}$ and $\frac{x^4(t)}{4} \le \frac{a^2}{2} + \frac{b^4}{4}$. This implies that $|x'(t)| \le \sqrt{2\cdot(\frac{a^2}{2}+\frac{b^4}{4})}$ and $|x(t)| \le \sqrt[4]{4\cdot(\frac{a^2}{2}+\frac{b^4}{4})}$. Thus the solution stays bounded.

(15) (a) Using the equation,

$$\begin{aligned} \frac{dx}{dt} &= -y + x^3 + xy^2 \\ \frac{dy}{dt} &= x + y^3 + x^2y \end{aligned},$$

we have $\frac{d}{dt}(x^2(t)+y^2(t)) = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$
$= 2x(-y+x^3+xy^2) + 2y(x+y^3+x^2y)$
$= -2xy + 2x^4 + 2x^2y^2 + 2xy + 2y^4 + 2x^2y^2$
$= 2x^4 + 4x^2y^2 + 2y^4 = 2(x^2+y^2)^2$.

(b) Let $r(t) = x^2(t)+y^2(t)$. From (a), we have $r'(t) = 2r^2$. Thus $r(t) = \frac{1}{-2t+\frac{1}{r_0}}$ where $r_0 = r(0) = x^2(0)+y^2(0)$.
Hence $\lim_{t\to\frac{1}{2r_0}^-} r(t) = \infty$.
(Hint: Let $r(t) = x^2(t)+y^2(t)$. Use the equation in (a) to find the explicit formula for $r(t)$.)

(16) Let $x_1(t) = y(t)$, $x_2(t) = y'(t)$ and $x_3(t) = y''(t)$. Then $\frac{dx_1}{dt} = y'(t) = x_2$, $\frac{dx_2}{dt} = y''(t) = x_3$, $\frac{dx_3}{dt} = y'''(t) = -by''(t) - cy'(t) - dy(t) = -dx_1 - cx_2 - bx_3$. Hence

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= x_3 \\ \frac{dx_3}{dt} &= -dx_1 - cx_2 - bx_3 \end{aligned}$$

(17) $L(2y'(t) - \int_0^t (t-\tau)^2 y(\tau) d\tau) = L(-2t)$

$\Rightarrow 2L(y'(t)) - 2s - L(\int_0^t (t-\tau)^2 y(\tau) d\tau)) = -\frac{2}{s^2}$

$\Rightarrow 2sL(y(t)) - 2 - L(t^2)L(y(t)) = -\frac{2}{s^2}$

$\Rightarrow 2sL(y(t)) - \frac{2}{s^3}L(y(t)) = -\frac{2}{s^2}$

$\Rightarrow (2s - \frac{2}{s^3})L(y(t)) = -\frac{2}{s^2} + 2$

$\Rightarrow (\frac{2s^4-2}{s^3})L(y(t)) = \frac{2s^2-2}{s^2}$

$\Rightarrow L(y(t)) = \frac{2s(s^2-1)}{2(s^4-1)} = \frac{s}{s^2+1}$

$\Rightarrow y(t) = L^{-1}(\frac{s}{s^2+1}) = \cos(t)$