Review Problems for Final Exam

Math 3860

The information about the final exam can be found at
http://www.math.utoledo.edu/∼mtsui/de05f/exam/final.html

(1) (a) Show that the substitution $v = ax + by + c$ transforms the differential equation $\frac{dy}{dx} = F(ax + by + c)$ into a separable equation.
(b) Show that the substitution $v = \frac{y}{x}$ transforms the differential equation $\frac{dy}{dx} = F(\frac{y}{x})$ into a separable equation.
(c) Show that the substitution $v = y^{1-n}$ transforms the Bernoulli equation $\frac{dy}{dx} + P(x)y = Q(x)y^n$ into the linear equation $\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$.

(2) Solve the following equations.
(a) $\frac{dy}{dt} = 6t(y-1)^{\frac{2}{3}}$.
(b) $t\frac{dy}{dt} - y = 2t^2 y$.
(c) $\frac{dy}{dt} + 2y = -2t + 1$.
(d) $\frac{dy}{dt} - 3y = \cos(2t)$.
(e) $\frac{dy}{dt} - 3y = -e^{-2t}$.
(f) $(t^2+1)\frac{dy}{dt} + 4ty = 4t$.
(g) $t\frac{dy}{dt} + 3y + t^2 = 0$.
(h) $(t+1)\frac{dy}{dt} + y = (t+1)^{\frac{3}{2}}$.
(i) $t\frac{dy}{dt} - 6y = 12t^4y^2$. (Bernoulli equation).
(j) $\frac{dy}{dx} = (x+y)^2$.
(k) $\frac{dy}{dx} = \frac{1}{(x+y)^2}$.
(l) $\frac{dy}{dx} = \frac{y - 2\sqrt{x^2+y^2}}{x}$.
(m) $\frac{dy}{dx} = \frac{x-y}{x+y}$.
(n) $\frac{dy}{dt} = \frac{y^3 - 6ty}{4y + 3t^2 - 3ty^2}$.

(3) What can you say about the behavior of the solution to
$\frac{dy}{dt} = y(3-y)(1+t^2+y^2)$, $y(0) = 2$? For example, the asymptotic behavior of the solution. Please justify your answer. (Don't try to solve the ODE.)

(4) In each problem, determine the equilibrium points, and classify each one as asymptotically stable, unstable, or semistable.
(a) $\frac{dy}{dt} = y^3 - 3y^2 + 2y$
(b) $\frac{dy}{dt} = (y^3 - 3y^2 + 2y)(y-3)^2$

(5) Find the solution of the following differential equations.
(a) $y''(t) + 5y'(t) + 6y(t) = 0$.
(b) $y''(t) + 4y'(t) + 4y(t) = 0$.
(c) $y''(t) + 4y'(t) + 8y(t) = 0$.

(d) $y^{(6)}(t) + 64y(t) = 0$.
(e) $(D^2 + 4D + 8)^2(D - 2)^3 D^2 y = 0$.
(f) $t^2 y''(t) + 2ty'(t) - 2y = 0$.
(g) $t^2 y''(t) + 5ty'(t) + 4y = 0$.
(h) $t^2 y''(t) + 5ty'(t) + 8y = 0$.
(i) $t^3 y'''(t) - 3ty'(t) + 3y = 0$.
(j) $y''(t) + 5y'(t) + 4y = g(t)$ with $y(0) = 0$ and $y'(0) = 0$ where

$$g(t) = \begin{cases} 0, & 0 \leq t < 2, \\ 3(t-2), & 2 \leq t < 4, \\ 6, & 3 \leq t. \end{cases}$$

(k) $y''(t) + 4y'(t) + 5y = g(t)$ with $y(0) = 0$ and $y'(0) = 0$ where

$$g(t) = \begin{cases} 0, & 0 \leq t < 2, \\ 1, & 2 \leq t < 4, \\ 0, & 3 \leq t. \end{cases}$$

(l) $y''(t) + 2y'(t) + 2y(t) = \delta(t-2)$, with $y(0) = 0$ and $y'(0) = 0$.
(m) $y''(t) - 4y'(t) + 3y(t) = \delta(t-2)$, with $y(0) = 1$ and $y'(0) = 1$.
(n) $y''(t) - 4y'(t) + 4y(t) = \delta(t-2)$, with $y(0) = 0$ and $y'(0) = 0$.

(6) In the following problem, a differential equation and one solution y_1 of the homogeneous equation are given. Use the method of reduction of order to find the general solution of the differential equation.
$ty''(t) - (1-2t)y'(t) + (t-1)y(t) = te^t; y_1(t) = e^t$.

(7) Find the general solution of the following differential equations.
(a) $y''(t) + 4y'(t) + 4y(t) = e^{-2t} + e^{3t}$
(b) $y''(t) + 4y'(t) + 4y(t) = e^{-2t} \ln t$
(c) $y''(t) + 9y = \sin(3t) + \cos(2t)$
(d) $y''(t) + 4y = \sec^2(2t)$
(e) $y''(t) + 5y'(t) + 6y(t) = 4e^{-2t} + e^{3t} + \sin(t)$
(f) $t^2 y''(t) - 4ty'(t) + 6y = t^3 + 1$

(8) Set up the appropriate form of a particular solution $y_p(t)$, but do not determine the values of the coefficients. For example, the particular solution of the equation $y''(t) + y(t) = \sin(t)$ is $y_p(t) = c_1 t \sin(t) + c_2 t \sin(t)$.
(a) $y''(t) - 5y'(t) + 6y(t) = te^{2t} + e^{3t} + e^{-2t}$
(b) $y''(t) + 4y = t\sin(2t) - 3\cos(t)$
(c) $y''(t) - 4y'(t) + 5y(t) = e^{2t}\sin(t) + e^{3t}\sin(t)$

(9) (a) Find the general solution of the equation
$y''(t) + y(t) = 2\cos(t)$

(b) Sketch the graph of the solution $y(t)$ to the equation $y''(t) + y(t) = 2\cos(t)$ with $y(0) = 0$ and $y'(0) = 0$. Also describe the behavior of the solution.

(c) Find the initial value a and b such that the solution to $y''(t) + y(t) = 2\cos(t)$ with $y(0) = a$ and $y'(0) = b$ stays bounded.

(10) Express the solution of the given initial value problem in terms of the convolution integral.

(a) $y''(t) + 4y'(t) + 5y(t) = e^{2t}\cos(t)$, with $y(0) = 0$ and $y'(0) = 0$.

(b) $y''(t) - 2y'(t) + y(t) = te^t$, with $y(0) = 0$ and $y'(0) = 0$.

(c) $y''(t) - 3y'(t) + 2y(t) = te^t + te^{2t}$, with $y(0) = 0$ and $y'(0) = 0$.

(11) Find the general solution of the given system of equations, describe the behavior of the solutions as $t \to \infty$. Also draw a few trajectories to indicate the behavior of the system.

(a)

$$\begin{aligned} \frac{dx_1}{dt} &= ax_1 + bx_2 \\ \frac{dx_2}{dt} &= bx_1 + ax_2 \end{aligned}$$

(b)

$$\begin{aligned} \frac{dx_1}{dt} &= ax_1 - bx_2 \\ \frac{dx_2}{dt} &= bx_1 + ax_2 \end{aligned}$$

(c)

$$\begin{aligned} \frac{dx_1}{dt} &= -5x_1 + 2x_2 \\ \frac{dx_2}{dt} &= -4x_1 + x_2 \end{aligned}$$

(d)

$$\begin{aligned} \frac{dx_1}{dt} &= -2x_1 - x_2 \\ \frac{dx_2}{dt} &= 2x_1 - 4x_2 \end{aligned}$$

(e)

$$\begin{aligned} \frac{dx_1}{dt} &= -5x_1 + 3x_2 \\ \frac{dx_2}{dt} &= -3x_1 + x_2 \end{aligned}$$

(f)

$$\begin{aligned} \frac{dx_1}{dt} &= 4x_1 + -x_2 \\ \frac{dx_2}{dt} &= x_1 + 2x_2 \end{aligned}$$

(12) Find the solution of the given initial value problem. You may use the result from 14d and 14e.

(a)

$$\begin{aligned} \frac{dx_1}{dt} &= -2x_1 - x_2, \quad x_1(0) = 1 \\ \frac{dx_2}{dt} &= 2x_1 - 4x_2, \quad x_2(0) = -1. \end{aligned}$$

(b)

$$\begin{aligned} \frac{dx_1}{dt} &= -5x_1 + 3x_2, \quad x_1(0) = 1 \\ \frac{dx_2}{dt} &= -3x_1 + x_2, \quad x_2(0) = -1. \end{aligned}$$

(13) Determine the stability of the following linear systems(You don't have to find the general solution). Note that the linear system is asymptotically stable if all the solutions converge, the linear system is stable if all the solutions remain bounded but do not converge and the linear system is unstable if some solutions diverge to infinity.

(a) Determine the stability of the system in (14c)-(14f).

(b)

$$\begin{aligned} \frac{dx_1}{dt} &= 5x_1 - 2x_2 \\ \frac{dx_2}{dt} &= 4x_1 - x_2 \end{aligned}$$

(c)

$$\begin{aligned} \frac{dx_1}{dt} &= 2x_1 + x_2 \\ \frac{dx_2}{dt} &= -2x_1 + 4x_2 \end{aligned}$$

(d)

$$\begin{aligned} \frac{dx_1}{dt} &= 2x_1 + 4x_2 \\ \frac{dx_2}{dt} &= -2x_1 - 2x_2 \end{aligned}$$

(14) Suppose $x(t)$ satisfies $x''(t) + x^3(t) = 0$, $x'(0) = a$ and $x(0) = b$.

(a) Show that $\frac{(x'(t))^2}{2} + \frac{x^4(t)}{4} = \frac{a^2}{2} + \frac{b^4}{4}$.
(Hint: Integrate the expression $x''(t)x'(t) + x^3(t)x'(t) = 0$.)

(b) Show that all the solutions to the equation $x''(t) + x^3(t) = 0$ stay bounded.

(15) (a) Suppose $(x(t), y(t))$ satisfies

$$\begin{aligned} \frac{dx}{dt} &= -y + x^3 + xy^2 \\ \frac{dy}{dt} &= x + y^3 + x^2y. \end{aligned}$$

Show that $\frac{d}{dt}(x^2(t) + y^2(t)) = (x^2(t) + y^2(t))^2$.

(b) Show that the equilibrium point $(0, 0)$ is unstable.
(Hint: Let $r(t) = x^2(t) + y^2(t)$. Use the equation in (a) to find the explicit formula for $r(t)$.)

(16) Show that the equation $y'''(t) + by''(t) + cy'(t) + dy(t) = 0$ can be written as a linear system in x_1, x_2 and x_3. where $x_1(t) = y(t)$ and $x_2(t) = y'(t)$ and $x_3(t) = y''(t)$.

(17) Find the solution of $2y'(t) - \int_0^t (t - \tau)^2 y(\tau) d\tau = -2t$ with $y(0) = 1$.