Solution to Review Problems for Midterm I

MATH 3860 – 001

Disclaimer: My solution is only correct up to a constant.
Please email(Mao-Pei.Tsui@math.utoledo.edu) me if you find any mistake.
You can skip problems 5j and 5k. They are from section 2.6 which is not a topic for this midterm.

(1) (a) From the slope field, it seems that all solutions converge to $y = x + 1$. One can also check easily that $y = x+1$ is a solution of $\frac{dy}{dx} = (x - y + 1)(1 + \sin(xy)) + 1$.

(b) One can check easily that $y = 0$ and $y = 2$ are solutions to $\frac{dy}{dx} = (2y - y^2)(1 + x^2y^2)$. Suppose $y(0) > 2$ or $0 < y(0) < 2$, we have $\lim_{x\to\infty} y(x) = 2$. Suppose $y(0) < 0$, we have $y(x) \to -\infty$.

(2) (a) Let $v = ax + by + c$. Since $\frac{dv}{dx} = a + b\frac{dy}{dx}$ and $\frac{dy}{dx} = F(ax + by + c)$, we have $\frac{dv}{dx} = a + bF(ax + by + c) = a + bF(v)$. Thus $\int \frac{dv}{a+bF(v)} = \int dx$.

(b) Let $v = x + y + 1$. Since $\frac{dv}{dx} = 1 + \frac{dy}{dx}$ and $\frac{dy}{dx} = (x + y + 1)^2 = v^2$, we have $\frac{dv}{dx} = 1 + v^2$. Thus $\int \frac{dv}{1+v^2} = \int dx$, $\arctan(v) = x + c$ and $v = \tan(x + c)$. Recall that $v = x + y + 1$, we have $y = v - x - 1 = \tan(x + c) - x - 1$.

(3) (a) Let $v = \frac{y}{x}$, i.e. $y = xv$. Using $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and $\frac{dy}{dx} = F(\frac{y}{x}) = F(v)$, we have $v + x\frac{dv}{dx} = F(v)$. It can be rewritten as $x\frac{dv}{dx} = F(v) - v$ which can be solved by $\int \frac{dv}{F(v)-v} = \int \frac{dx}{x}$.

(b) The equation $x^2\frac{dy}{dx} = y^2 + xy - x^2$ can be simplified as $\frac{dy}{dx} = \frac{y^2+xy-x^2}{x^2} = (\frac{y}{x})^2 + \frac{y}{x} - 1$. Let $v = \frac{y}{x}$. Note that $(\frac{y}{x})^2 + \frac{y}{x} - 1 = v^2 + v - 1$. Following the same computation as above, we have $x\frac{dv}{dx} = v^2 - 1$. It can be solved by $\int \frac{dv}{v^2-1} = \int \frac{dx}{x}$. Note that $\int \frac{dv}{v^2-1} = \int \frac{1}{2(v-1)} - \frac{1}{2(v+1)}dv = \frac{1}{2}\ln|\frac{v-1}{v+1}| + D$. Hence $\frac{1}{2}\ln|\frac{v-1}{v+1}| = \ln|x| + C$ and $\ln|\frac{v-1}{v+1}| = 2\ln|x| + c = \ln|x|^2 + c$. Therefore $\frac{v-1}{v+1} = Cx^2$ and $v = \frac{1+Cx^2}{1-Cx^2}$. Recall that $v = \frac{y}{x}$. It follows that $y = xv = x \cdot \frac{1+Cx^2}{1-Cx^2}$.

(4) (a) Separate the equation, we have $\int(3y^2 - 6y)dy = \int 2xdx$ and $y^3 - 3y^2 = x^2 + C$. Plugging $y(0) = 1$, we have $C = -2$ Thus the solution (x, y) satisfies $y^3 - 3y^2 = x^2 - 2$. From the equation $y' = \frac{2x}{3y^2-6y}$, we know that the solution y will not exist if $3y^2 - 6y = 3y(y - 2) = 0$, that is $y = 0$ and $y = 2$. If $y = 0$, we have $0 = x^2 - 2$, i.e. $x = \pm\sqrt{2}$. From the graph in Figure 3, we know that the solution exists on the interval $-\sqrt{2} < x < \sqrt{2}$.

(b) Using $y^3 - 3y^2 = x^2 + C$ and $y(\sqrt{18}) = 4$, we have $64 - 48 = 18 + C$ and $C = -2$. So the solution (x, y) satisfies $y^3 - 3y^2 = x^2 - 2$. From the graph of $y^3 - 3y^2 = x^2 - 2$, we know that the solution exists on the interval $-\infty < x < \infty$.

(5) Solve the following equations.

(a) $y(t) = \frac{2t}{3} + \frac{1}{9} + Ce^{-3t}$.Thus $\lim_{t\to\infty} y(t) - \frac{2t}{3} - \frac{1}{9} = 0$.

(b) $y(t) = -\frac{1}{2}\cos(t) + \frac{1}{2}\sin(t) + (y_0 + \frac{1}{2})e^t$ If $y_0 = -\frac{1}{2}$ then the solution remains finite as $t \to \infty$. We have used the fact that $\lim_{t\to\infty} e^{3t} = \infty$.

(c) $y(t) = \frac{1}{4}e^{-t} + (y_0 - \frac{1}{4})e^{3t}$. Note that $\lim_{t\to\infty} e^{3t} = \infty$ and $\lim_{t\to\infty} e^{-t} = 0$. The condition $y_0 < \frac{1}{4}$ will imply that $\lim_{t\to\infty} y(t) = -\infty$.

(d) $y(t) = 2 + \frac{C}{(t^2+1)^{\frac{3}{2}}}$. Hence $\lim_{t\to\infty} y(t) = 2$.

(e) $y(t) = t^3 \ln|t| + Ct^3$.

(f) $y(t) = \frac{2e^t + C}{t}$.

(g) $y(t) = \frac{(t^2+t)}{\sqrt{2t+1}} + \frac{C}{\sqrt{2t+1}}$. Note that the integrating factor is $\mu(t) = e^{\int \frac{1}{2t+1}dt} = e^{\frac{1}{2}\ln|2t+1|} = e^{\ln|2t+1|^{\frac{1}{2}}} = |2t+1|^{\frac{1}{2}}$.

(h) Let $v = y^{1-\frac{2}{3}} = y^{\frac{1}{3}}$. Then we have $v' - \frac{2}{t}v = 4t^3$ and $v(t) = 2t^4 + Ct^2$. Using $y = v^3$, we get $y = v^3 = (2t^4 + Ct^2)^3$.

(i) This can be rewritten as $\frac{dy}{dx} = 5x^4y^2 - 4x^2y^2 = y^2(5x^4 - 4x^2)$. The general solution is $y = \frac{-1}{x^5 - \frac{4}{3}x^3 + C}$.

(j) Let $P = 6xy + 2y^2$ and $Q = 9x^2 + 8xy + y$. Note that $P_y = 6x + 4y$ and $Q_x = 18x + 8y$. This equation is not exact. Let $\mu = \mu(x, y)$ be the integrating factor. $\mu(6xy + 2y^2) + \mu(9x^2 + 8xy + y)\frac{dy}{dx} = 0$ is exact if $(\mu(6xy + 2y^2))_y = (\mu(9x^2+8xy+y))_x$, i.e. $\mu_y(6xy+2y^2)+\mu(6x+4y) = \mu_x(9x^2+8xy+y)+\mu(18x+8y)$. This can be simplified as $\mu_y 2y(3x+y) - \mu_x(9x^2+8xy+y) = 4\mu(3x+y)$ Hence there exists $\mu = \mu(y)$ which is a solution of $\mu_y 2y(3x+y)) = 4\mu(3x+y)$. This is the same as $\frac{d\mu}{dy} = \frac{2\mu}{y}$. Thus $\mu = y^2$. Now $y^2(6xy + 2y^2) + y^2(9x^2 + 8xy + y)\frac{dy}{dx} = 0$ Solving $F_x = y^2(6xy+2y^2) = 6xy^3+2y^4$ and $F_y = y^2(9x^2+8xy+y) = 9x^2y^2+8xy^3+y^3$. We have $F(x, y) = 3x^2y^3 + 2xy^4 + \frac{y^4}{4} = C$.

(k) This is a exact equation. We have $xe^y + \sin(x)y + e^y = C$.

(l) There is a typo in this problem. This problem should be $\frac{dy}{dx} = (x + y)^2 - 1$. Let $v = x + y$. We have $\frac{dv}{dx} = 1 + \frac{dy}{dx} = 1 + (x + y)^2 - 1 = v^2$ Thus $\int \frac{dv}{v^2} = \int dx$, $-\frac{1}{v} = x + c$ and $v = -\frac{1}{x+c}$. Recall that $v = x + y$. Hence $y + x = -\frac{1}{x+c}$ and $y = -x - \frac{1}{x+c}$.

(m) The equation $2x^2y - x^3\frac{dy}{dx} = y^3$ can be written as a homogeneous equation $\frac{dy}{dx} = \frac{-y^3+2x^2y}{x^3} = -(\frac{y}{x})^3 + 2\frac{y}{x}$. Let $v = \frac{y}{x}$. After simplification, we have $\int \frac{dv}{-v^3+v} = \int dx$. By partial fraction, we have $\frac{1}{-v^3+v} = \frac{1}{v} - \frac{1}{2(v-1)} - \frac{1}{2(v+1)}$ and $\int \frac{dv}{-v^3+v} = \ln|v| - \frac{1}{2}\ln|v^2 - 1| + C_1$. We have $\ln|v| - \frac{1}{2}\ln|v^2 - 1| = x + C$. Substituting $v = \frac{y}{x}$, $\ln|\frac{y}{x}| - \frac{1}{2}\ln|\frac{y^2}{x^2} - 1| = x + C$

(n) The equation $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$ can be written as a separable equation $\frac{dy}{dx} = 1 + x^2 + y^2(1 + x^2) = (1 + x^2)(1 + y^2)$. Solving $\int \frac{dy}{1+y^2} = \int(1 + x^2)dx$, we get $\arctan(y) = x + \frac{x^3}{3} + C$ and $y = \tan(x + \frac{x^3}{3} + C)$.

(6) Using separation of variables, we have $y(t) = (\frac{1}{-3t+\frac{1}{y_0^3}})^{\frac{1}{3}}$. Note that $\frac{1}{3y_0^3} > 0$. We have $\lim_{t\to(\frac{1}{3y_0^3})^-} y(t) = \infty$.

(7) Obviously, $y_1(t) = 0$ is a solution of $\frac{dy}{dt} = \frac{y\sin(x)}{1+t^2+y^2}$. Now we have $y(0) = -2 < 0 = y_1(0)$. Obviously, the solution to $\frac{dy}{dt} = \frac{y\sin(x)}{1+t^2+y^2}$ is unique. We have $y(t) < y_1(0) = 0$, i.e. y is always negative.

(8) The equation $(9-t^2)\frac{dy}{dt} + \frac{y}{t} = \cos(t)$ can be rewritten as $\frac{dy}{dt} + \frac{1}{t(3-t)(3+t)}y = \frac{\cos(t)}{(3-t)(3+t)}$. Both $\frac{1}{t(3-t)(3+t)}$ and $\frac{\cos(t)}{(3-t)(3+t)}$ are continuous on $(-\infty,-3)\cup(-3,0)\cup(0,3)\cup(0,\infty)$.
   (a) Since $-1 \in (-3,0)$, this solution exists on the interval $(-3,0)$.
   (b) Since $1 \in (0,3)$, this solution exists on the interval $(0,3)$.
   (c) Since $4 \in (3,\infty)$, this solution exists on the interval $(3,\infty)$.
   (d) Since $-4 \in (-\infty,-3)$, this solution exists on the interval $(-\infty,-3)$.

(9) In each problem, determine the equilibrium points, and classify each one as asymptotically stable, unstable, or semistable.
   (a) The equilibrium points are $y = k\pi$ where k is an integer. Since $y^2\sin^2(y) > 0$ if $y \neq k\pi$, we know that all the equilibrium points are semistable.
   (b) The equilibrium points are $y = k\pi$ where k is an integer. The sign graph of $y\sin(y)$ and $\sin(y)$ is the same when $y > 0$. The sign graph of $y\sin(y)$ and $-\sin(y)$ is the same when $y < 0$. Thus $y\sin(y) > 0$ when $2k\pi < y < (2k+1)\pi$ or $-(2k+1)\pi < y < -2k\pi$ where k is an nonnegative integer. Thus $y\sin(y) < 0$ when $(2k+1)\pi < y < (2k+2)\pi$ or $-(2k+2)\pi < y < -(2k+1)\pi$ where k is an nonnegative integer. Thus 0 is an semistable equilibrium point. $\{\pi, 3\pi, 5\pi, \cdots\}$ and $\{-2\pi, -4\pi, -6\pi, \cdots\}$ are asymptotically stable equilibrium points. $\{2\pi, 4\pi, 6\pi, \cdots\}$ and $\{-\pi, -3\pi, -5\pi, \cdots\}$ are unstable equilibrium points.
   (c) Let $f(y) = (-y^3+3y^2-2y)(y-3)^2$. We have $f(y) = -y(y^2-3y+2)(y-3)^2 = -y(y-1)(y-2)(y-3)^2$. Thus $f(y) > 0$ when $y \in (-\infty,0)\cup(1,2)$ and $f(y) < 0$ when $y \in (0,1)\cup(2,3)\cup(3,\infty)$. Therefore $\{0,2\}$ are are asymptotically stable equilibrium points, 1 is an unstable equilibrium point and 3 is a semistable equilibrium point.
   (d) Let $f(y) = y^3-3y^2+2y = y(y-1)(y-2)$. Thus $f(y) > 0$ when $y \in (0,1)\cup(2,\infty)$ and $f(y) < 0$ $y \in (-\infty,0)\cup(1,2)$. Hence $y = 1$ are are asymptotically stable equilibrium points, $y = 0$ and $y = 2$ are unstable equilibrium points.

(10) (a) Let $f(y) = \frac{y^2(y-2)}{y-1}$. We have $f(y) > 0$ if $y \in (0,1)\cup(2,\infty)\cup(-\infty,0)$ and $f(y) < 0$ if $y \in (1,2)$. Note that $y = 0$ and $y = 2$ are equilibrium solutions. The graph of y has a vertical tangent line when $y = 1$. Let $y(t)$ be the solution to $\frac{dy}{dt} = \frac{y^2(y-2)}{y-1}$ with $y(0) = y_0$. If $y_0 > 2$, $y(t)$ is increasing and it will escape

to ∞. If $1 < y_0 < 2$, $y(t)$ is will decrease to 1 in a finite time with a vertical tangent line when $y = 1$. If $0 < y_0 < 1$, $y(t)$ will increase to 1 in a finite time with vertical a vertical tangent line when $y = 1$. If $y_0 < 0$, $y(t)$ is increasing and $\lim_{t\to\infty} y(t) = 0$.

(b) $\frac{dy}{dt} = \frac{(y^2-4)}{y-1}$ Let $f(y) = \frac{(y^2-4)}{y-1}$. We have $f(y) > 0$ if $y \in (-2,1) \cup (2,\infty)$ and $f(y) < 0$ if $y \in (1,2) \cup (-\infty,-2)$. Note that $y = -2$ and $y = 2$ are equilibrium solutions. The graph of y has a vertical tangent line when $y = 1$. Let $y(t)$ be the solution to $\frac{dy}{dt} = \frac{(y^2-4)}{y-1}$ with $y(0) = y_0$. If $y_0 > 2$, $y(t)$ is increasing and it will escape to ∞. If $1 < y_0 < 2$, $y(t)$ is will decrease to 1 in a finite time with a vertical tangent line when $y = 1$. If $-2 < y_0 < 1$, $y(t)$ will increase to 1 in a finite time with vertical a vertical tangent line when $y = 1$. If $y_0 < -2$, $y(t)$ is decreasing and it will escape to $-\infty$.

(11) Since $\frac{dy}{dt} = 4y - y^2$, we have $\frac{d^2y}{dt^2} = \frac{d}{dt}(y') = \frac{d}{dt}(4y - y^2) = (4y' - 2yy') = (4-2y)y' = (4-2y)(4y-y^2) = 2y(1-y)(2-y)$.
Since $\frac{d^2y}{dt^2} = 2y(1-y)(2-y)$, we have $\frac{d^2y}{dt^2} > 0$ if $y \in (0,1) \cup (2,\infty)$ and $\frac{d^2y}{dt^2} < 0$ if $y \in (-\infty,0) \cup (1,2)$. Note that $\frac{dy}{dt} > 0$ if $y \in (0,4)$ and $\frac{dy}{dt} < 0$ if $y \in (-\infty,0) \cup (4,\infty)$.