Solution to Review Problems for Midterm II

MATH 3860 – 001

Correction:
(4i) should be $t^2y''(t) - t(t+2)y'(t) + (t+2)y(t) = t^4e^t(1+t)$.
Given that $y_1(t) = te^t$ is a solution of $t^2y''(t) - t(t+2)y'(t) + (t+2)y(t) = 0$.
You should do problems involving the exact equation. Here is one example.
Solve $(2xy^3 + y^4 + 1) + (3x^2y^2 + 4xy^3 + 3y^2)\frac{dy}{dx} = 0$.
Solution: Let $M(x,y) = 2xy^3 + y^4 + 1$ and $N(x,y) = 3x^2y^2 + 4xy^3 + 3y^2$, We have have $M_y = 6xy^2 + 4y^3$ and $N_x = 6xy^2 + 4y^3$. So $M_y = N_x$ and the given equation is exact. Thus there is a function $\phi(x,y)$ such that $\phi_x = M = 2xy^3+y^4+1$ and $\phi_y = N = 3x^2y^2+4xy^3+3y^2$. Integrating the first equation, we have $\phi(x,y) = \int(2xy^3+y^4+1)dx = x^2y^3+xy^4+x+h(y)$. Using $\phi_y = N = 3x^2y^2 + 4xy^3 + 3y^2$, we have $\frac{\partial(x^2y^3+xy^4+x+h(y))}{\partial y} = 3x^2y^2 + 4xy^3 + 3y^2$, $3x^2y^2 + 4xy^3 + h'(y) = 3x^2y^2 + 4xy^3 + 3y^2$, $h'(y) = 3y^2$ and $h(y) = \int(3y^2)dy = y^3 + c$. Hence $\phi(x,y) = x^2y^3 + xy^4 + x + h(y) = x^2y^3 + xy^4 + x + y^3$ and the solution satisfies $\phi(x,y) = x^2y^3 + xy^4 + x + y^3 = c$.

(1) Find the general solution of the following differential equations.
   (a) $y''(t) + 6y'(t) + 9y = 0$. The characteristic equation of $y''(t) + 6y'(t) + 9y(t) = 0$ is $r^2 + 6r + 9 = (r+3)^2 = 0$. We have repeated roots $r = -3$. Thus the general solution is $y(t) = c_1e^{-3t} + c_2te^{-3t}$.
   (b) $y''(t) + 5y'(t) + 4y = 0$. The characteristic equation of $y''(t) + 5y'(t) + 4y = 0$ is $r^2 + 5r + 4 = (r+1)(r+4) = 0$. We have $r = -1$ or $r = -4$. Thus the general solution is $y(t) = c_1e^{-t} + c_2e^{-4t}$.
   (c) $y''(t) + 4y'(t) + 5y = 0$. The characteristic equation of $y''(t) + 4y'(t) + 5y = 0$ is $r^2 + 4r + 5 = 0$. We have $r = -2 \pm i$. Note that $e^{(-2+i)t} = e^{-2t}e^{it} = e^{-2t}\cos(t) + ie^{-2t}\sin(t)$. Thus the general solution is $y(t) = c_1e^{-2t}\cos(t) + c_2e^{-2t}\sin(t)$.
   (d) $t^2y''(t) + 7ty'(t) + 8y(t) = 0$. Suppose $y(t) = t^r$, we have $y'(t) = rt^{r-1}$ and $y''(t) = r(r-1)t^{r-2}$. Thus $t^2y''(t) + 7ty'(t) + 8y(t) = (r(r-1) + 7r + 8)t^r = (r^2 + 6r + 8)t^r$. Thus $y = t^r$ is a solution of $t^2y''(t) + 7ty'(t) + 8y(t) = 0$ if $r^2 + 6r + 8 = (r+2)(r+4) = 0$. The roots of $r^2 + 6r + 8 = 0$ are -2 and -4. Therefore the general solution is $y(t) = c_1t^{-2} + c_2t^{-4}$.
   (e) $t^2y''(t) + 7ty'(t) + 10y(t) = 0$. Suppose $y(t) = t^r$, we have $y'(t) = rt^{r-1}$ and $y''(t) = r(r-1)t^{r-2}$. Thus $t^2y''(t) + 7ty'(t) + 10y(t) = (r(r-1) + 7r + 10)t^r = (r^2 + 6r + 10)t^r$. Thus $y = t^r$ is a solution of $t^2y''(t) + 7ty'(t) + 10y(t) = 0$ if $r^2 + 6r + 10 = 0$. The roots of $r^2 + 6r + 10 = 0$ are $-3 + i$ and $-3 - i$. Note that $t = e^{\ln t}$ and $t^{-3+i} = t^{-3}e^{i\ln t} = t^{-3}\cos(\ln t) + it^{-3}\sin(\ln t)$. Therefore the general solution is $c_1t^{-3}\cos(\ln t) + c_2t^{-3}\sin(\ln t)$.

(f) $t^2y''(t)+5ty'(t)+4y(t)=0$. Suppose $y(t)=t^r$, we have $y'(t)=rt^{r-1}$ and $y''(t)=r(r-1)t^{r-2}$. Thus $t^2y''(t)+5ty'(t)+4y(t)=(r(r-1)+5r+4)t^r=(r^2+4r+4)t^r$. Thus $y=t^r$ is a solution of $t^2y''(t)+5ty'(t)+4y(t)=0$ if $r^2+4r+4=(r+2)^2=0$. The roots of $r^2+4r+4=0$ are -2. Therefore the general solution is $c_1t^{-2}+c_2t^{-2}\ln t)$.

(g) $t^2y''(t)+ty'(t)+9y=0$. Suppose $y(t)=t^r$, we have $y'(t)=rt^{r-1}$ and $y''(t)=r(r-1)t^{r-2}$. Thus $t^2y''(t)+ty'(t)+9y(t)=(r(r-1)+r+9)t^r=(r^2+9)t^r$. Thus $y=t^r$ is a solution of $t^2y''(t)+\alpha ty'(t)+\beta y(t)=0$ if $r^2+9=0$. The roots of $r^2+9=0$ are $3i$ and $-3i$. Note that $t=e^{\ln t}$ and $t^{3i}=e^{i3\ln t}=\cos(3\ln t)+i\sin(3\ln t)$. Therefore the general solution is $c_1\cos(3\ln t)+c_2\sin(3\ln t)$.

(h) Let $p(t)=y'(t)$. Then $ty''(t)+2y'(t)=t^3+t^2+1$ can be rewritten as $tp'(t)+2p'(t)=t^3+t^2+1$. Thus $(t^2p(t))'=t^4+t^3+t$. Thus $p(t)=\frac{t^3}{5}+\frac{t^2}{4}+\frac{1}{2}+\frac{C}{t^2}$ and $y(t)=\int p(t)dt=\frac{t^4}{20}+\frac{t^3}{12}+\frac{t}{2}-\frac{C}{t}+D$.

(i) Let $p(t)=y'(t)$. Then $y''(t)+(y'(t))^3=0$ can be written as $p'(t)+p^3(t)=0$. We have $p(t)=\pm\frac{1}{\sqrt{2t+C}}$ and $y(t)=\int p(t)dt=\pm\int\frac{1}{\sqrt{2t+C}}dt=\pm\sqrt{2t+C}+D$.

(j) Let $p(t)=y'(t)$. Then $y''(t)=(t+y'(t))^2-1$ can be written as $p'(t)=(t+p(t))^2-1$. Let $v=t+p(t)$. We have $v'(t)=1+p'(t)$ and $p'(t)=v'(t)-1$. Thus $p'(t)=(t+p(t))^2-1$ is equivalent to $v'(t)-1=v^2(t)-1$, that is $v'(t)=v^2(t)$. We get $v(t)=\frac{1}{-t+C}$ and $y'(t)=p(t)=v(t)-t=\frac{1}{-t+C}-t$. Hence $y(t)=\int(\frac{1}{-t+C}-t)dt=-\ln|-t+C|-\frac{t^2}{2}+D$.

(2) Find the solution of the following initial value problems.

(a) $y''(t)+4y'(t)+5y=0$, $y(0)=1$ and $y'(0)=3$. From (1C), we have $y(t)=c_1e^{-2t}\cos(t)+c_2e^{-2t}\sin(t)$ and $y'(t)=-2c_1e^{-2t}\cos(t)-c_1e^{-2t}\sin(t)-2c_2e^{-2t}\sin(t)+c_2e^{-2t}\cos(t)=(-2c_1+c_2)e^{-2t}\cos(t)+(-c_1-2c_2)e^{-2t}\sin(t)$. Using $y(0)=1$ and $y'(0)=3$, we have $c_1=1$ and $-2c_1+c_2=3$. So $c_1=1$ and $c_2=5$. Hence $y(t)=e^{-2t}\cos(t)+5e^{-2t}\sin(t)$.

(b) $t^2y''(t)+7ty'(t)+10y(t)=0$, $y(1)=2$ and $y'(1)=-5$. From (1C), we have $y(t)=c_1t^{-3}\cos(\ln t)+c_2t^{-3}\sin(\ln t)$ and $y'(t)=-3c_1t^{-4}\cos(\ln t)-c_1t^{-3}\frac{\sin(\ln t)}{t}-3c_2t^{-4}\sin(\ln t)+c_2t^{-3}\frac{\cos(\ln t)}{t}=(-3c_1+c_2)t^{-4}\cos(\ln t)+(-c_1-3c_2)t^{-4}\sin(\ln t)$. Using $y(1)=2$ and $y'(1)=-5.$, we have $c_1=2$ and $-3c_1+c_2=-5$. So $c_1=2$ and $c_2=1$. Hence $y(t)=2t^{-3}\cos(\ln t)+t^{-3}\sin(\ln t)$.

(3) In the following problems, a differential and one solution y_1 are given. Use the method of reduction of order to find the general solution solution.

(a) $t^2y''(t)-t(t+2)y'(t)+(t+2)y(t)=0; y_1(t)=t$. Rewrite the equation $t^2y''(t)-t(t+2)y'(t)+(t+2)y(t)=0$ as $y''-\frac{t+2}{t}y'+\frac{t+2}{t^2}y=0$. So $p(t)=-\frac{t+2}{t}$. Let y_2 be another solution of $t^2y''(t) - t(t+2)y'(t) + (t+2)y(t) = 0$. We have $(\frac{y_2}{y_1})' = \frac{y_1y_2' - y_1'y_2}{y_1^2} = \frac{W(t)}{y_1^2} = \frac{Ce^{-\int p(t)dt}}{t^2} = \frac{Ce^{\int \frac{t+2}{t}dt}}{t^2} = \frac{Ce^{\int (1+\frac{2}{t})dt}}{t^2} = \frac{Ce^{(t+2\ln(t))}}{t^2} = \frac{Ce^te^{2\ln t}}{t^2} = \frac{Ce^tt^2}{t^2} = Ce^t$. So $\frac{y_2}{y_1} = \int Ce^tdt = Ce^t + D$ and $y_2 = y_1(Ce^t + D) = t(Ce^t + D) = Cte^t + Dt$. So the general solution is $y = Cte^t + Dt$.

(b) $(t+1)y''(t) - (t+2)y'(t) + y(t) = 0; y_1(t) = e^t$. Rewrite the equation $(t+1)y''(t) - (t+2)y'(t) + y(t) = 0$ as $y'' - \frac{t+2}{t+1}y' + \frac{1}{t+1}y = 0$. So $p(t) = -\frac{t+2}{t+1}$. Let y_2 be another solution of $(t+1)y''(t) - (t+2)y'(t) + y(t) = 0$. We have $(\frac{y_2}{y_1})' = \frac{y_1y_2' - y_1'y_2}{y_1^2} = \frac{W(t)}{y_1^2} = fracCe^{-\int p(t)dt}e^{2t} = \frac{Ce^{\int \frac{t+2}{t+1}dt}}{t^2} = \frac{Ce^{\int (1+\frac{1}{t+1})dt}}{e^{2t}} = \frac{Ce^{(t+\ln(t+1))}}{e^{2t}} = \frac{Ce^te^{\ln(t+1)}}{e^{2t}} = \frac{Ce^t(t+1)}{e^{2t}} = Ce^{-t}(t+1) = C(te^{-t} + e^{-t})$. So $\frac{y_2}{y_1} = \int C(te^{-t} + e^{-t})dt = C(-te^{-t} - 2e^{-t}) + D$ and $y_2 = y_1(C(-te^{-t} - 2e^{-t}) + D) = e^t(C(-te^{-t} - 2e^{-t}) + D) = -C(t+2) + De^t$. So the general solution is $y = c(t+2) + de^t$.

(4) Find the general solution of the following differential equations.

(a) $y''(t) + 5y'(t) + 6y(t) = e^t + \sin(t)$.
Solving $r^2 + 5r + 6 = (r+2)(r+3) = 0$, we know that the solution of $y''(t) + 5y'(t) + 6y(t) = 0$ is $y(t) = c_1e^{-2t} + c_2e^{-3t}$. We try $y_p = ce^t + d\sin(t) + e\cos(t)$ to be a particular solution of $y''(t) + 5y'(t) + 6y(t) = e^t + \sin(t)$. We have $y_p = ce^t + d\sin(t) + e\cos(t)$, $y_p' = ce^t + d\cos(t) - e\sin(t)$, $y_p'' = ce^t - d\sin(t) - e\cos(t)$ and $y_p''(t) + 5y_p'(t) + 6y_p(t) = (ce^t - d\sin(t) - e\cos(t)) + 5(ce^t + d\cos(t) - e\sin(t)) + 6(ce^t + d\sin(t) + e\cos(t)) = (c + 5c + 6c)e^t + (-d - 5e + 6d)\sin(t) + (-e + 5d + 6e)\cos(t) = 12ce^t + (5d - 5e)\sin(t) + (5d + 5e)\cos(t) = e^t + \sin(t)$ if $12c = 1$, $5d - 5e = 1$ and $5d + 5e = 0$. So $c = \frac{1}{12}$, $d = \frac{1}{10}$ and $e = -\frac{1}{10}$. Thus the general solution of $y''(t) + 5y'(t) + 6y(t) = e^t + \sin(t)$ is $y(t) = \frac{1}{12}e^t + \frac{1}{10}\sin(t) + \frac{1}{10}\cos(t) + c_1e^{-2t} + c_2e^{-3t}$.

(b) $y''(t) + 4y = 2\sin(2t) + 3\cos(t)$ Solving $r^2 + 4 = 0$, we know that the solution of $y''(t) + 4y = 0$ is $y(t) = c_1\sin(2t) + c_2\cos(2t)$. We try $y_p = ct\sin(2t) + dt\cos(2t) + e\sin(t) + f\cos(t)$ to be a particular solution of $y''(t) + 4y = 2\sin(2t) + 3\cos(t)$. We have $y_p = ct\sin(2t) + dt\cos(2t) + e\sin(t) + f\cos(t)$,
$y_p' = c\sin(2t) + 2ct\cos(2t) + d\cos(2t) - 2dt\sin(2t) + e\cos(t) - f\sin(t)$,
$y_p'' = 4c\cos(2t) - 4ct\sin(2t) - 4d\sin(2t) - 4dt\cos(2t) - e\sin(t) - f\cos(t)$ and
$y_p''(t) + 4y_p(t) = 4c\cos(2t) - 4d\sin(2t) + 3e\sin(t) + 3f\cos(t) = 2\sin(2t) + 3\cos(t)$ if $c = 0$, $d = -\frac{1}{2}$, $e = 0$ and $f = 1$. Thus the general solution of $y''(t) + 4y = 2\sin(2t) + 3\cos(t)$ is $y(t) = -\frac{1}{2}t\cos(2t) + \cos(t) + c_1\sin(2t) + c_2\cos(2t)$.

(c) $y''(t) + 4y = 4e^{4t}$ We try $y_p(t) = ce^{4t}$. Then $y_p' = 4ce^{4t}$, $y_p'' = 16ce^{4t}$. So $y_p''(t) + 4y_p = 20ce^{4t} = 4e^{4t}$ if $c = \frac{1}{5}$. The general solution is $y(t) = \frac{1}{5}e^{4t} + c_1\sin(2t) + c_2\cos(2t)$.

(d) $y''(t)+4y'+4y(t)=e^{-2t}+e^{2t}$ Solving $r^2+4r+4=(r+2)^2=0$, we know that the solution of $y''(t)+4y+4y(t)=0$ is $y(t)=c_1e^{-2t}+c_2te^{-2t}$. We try $y_p=ct^2e^{-2t}+de^{2t}$ to be a particular solution of $y''(t)+4y+4y(t)=e^{-2t}+e^{2t}$. We have $y_p=ct^2e^{-2t}+de^{2t}$, $y_p'=2cte^{-2t}-2ct^2e^{-2t}+2de^{2t}$, $y_p''=2ce^{-2t}-4cte^{-2t}-4cte^{-2t}+4ct^2e^{-2t}+4de^{2t}=2ce^{-2t}-8cte^{-2t}+4ct^2e^{-2t}+4de^{2t}$ and $y_p''(t)+4y_p'(t)+4y_p(t)=(2ce^{-2t}-8cte^{-2t}+4ct^2e^{-2t}+4de^{2t})+4(2cte^{-2t}-2ct^2e^{-2t}+2de^{2t})+4(ct^2e^{-2t}+de^{2t})=2ce^{-2t}+16de^{2t}$. So $y_p''(t)+4y_p'(t)+4y_p(t)=e^{-2t}+e^{2t}$ if $2c=1$ and $16d=1$, $c=\frac{1}{2}$ and $d=\frac{1}{16}$. Thus the general solution of $y''(t)+4y'+4y(t)=e^{-2t}+e^{2t}$ is $y(t)=\frac{1}{2}t^2e^{-2t}+\frac{1}{16}e^{2t}+c_1e^{-2t}+c_2te^{-2t}$.

(e) $y''(t)+5y'(t)+6y(t)=t^2+1$. Solving $r^2+5r+6=(r+2)(r+3)=0$, we know that the solution of $y''(t)+5y+6y(t)=0$ is $y(t)=c_1e^{-2t}+c_2e^{-3t}$. We try $y_p(t)=at^2+bt+c$ to be a particular solution of $y''(t)+5y'(t)+6y(t)=t^2+1$. So $y_p'=2at+b$, $y_p''=2a$ and $y_p''(t)+5y_p'(t)+6y_p(t)=2a+5(2at+b)+6(at^2+bt+c)=6at^2+(10a+6b)t+2a+5b+6c$. So $y_p''(t)+5y_p'(t)+6y_p(t)=t^2+1$ if $6a{=}1$, $10a+6b=0$ and $2a+5b+6c=1$. Thus $a=\frac{1}{6}$, $b=-\frac{5}{3}a=-\frac{5}{18}$ and $c=\frac{1-2a-5b}{6}=\frac{1-\frac{1}{3}+\frac{25}{18}}{6}=\frac{\frac{18-6+25}{18}}{6}=\frac{37}{108}$. The general solution of $y''(t)+5y'(t)+6y(t)=t^2+1$ is $y(t)=\frac{1}{6}t^2-\frac{5}{18}t+\frac{37}{108}+c_1e^{-2t}+c_2e^{-3t}$.

(variation of parameter) Suppose $y_1(t)$ and $y_2(t)$ are independent solutions of $y''(t)+p(t)y'(t)+q(t)y(t)=0$. Then a particular solution is given by
$y_p(t)=-y_1(t)\int\frac{y_2(t)g(t)}{W(t)}dt+y_2(t)\int\frac{y_2(t)g(t)}{W(t)}dt$
where $W(t)=W(y_1,y_2)(t)=y_1(t)y_2'(t)-y_2(t)y_1'(t)$ is the Wronskian of y_1 and y_2.

(f) $t^2y''(t)-ty'(t)-3y(t)=4t^2$.
First, we solve $t^2y''(t)-ty'(t)-3y(t)=0$. Suppose $y(t)=t^r$, we have $y'(t)=rt^{r-1}$ and $y''(t)=r(r-1)t^{r-2}$. Thus $t^2y''(t)-ty'(t)-3y(t)=(r(r-1)-r-3)t^r=(r^2-2r-3)t^r$. Thus $y=t^r$ is a solution of $t^2y''(t)-ty'(t)-3y(t)=0$ if $r^2-2r-3=(r-3)(r+1)=0$. The roots of $r^2-2r-3=0$ are -1 and 3. Therefore the general solution is $c_1t^{-1}+c_2t^3$.
Let $y_1=t^{-1}$ and $y_2=t^3$ to be the solutions of $t^2y''(t)-ty'(t)-3y(t)=0$. $W(y_1,y_2)(t)=y_1(t)y_2'(t)-y_2(t)y_1'(t)=t^{-1}\cdot(3t^2)-t^3\cdot(-1t^{-2})=4t$.
Now $g(t)=4t^2$. We have
$\int\frac{y_2g(t)}{W(y_1,y_2)(t)}dt=\int\frac{t^34t^2}{4t}dt=\int t^4dt=\frac{1}{5}t^5+c$ and $\int\frac{y_1g(t)}{W(y_1,y_2)(t)}dt=\int\frac{t^{-1}4t^2}{4t}dt=\int 1dt=t+c$. So $y(t)=-y_1(t)\int\frac{y_2(t)g(t)}{W(t)}dt+y_2(t)\int\frac{y_2(t)g(t)}{W(t)}dt=-t^{-1}(\frac{1}{5}t^5+d)+t^3(t+c)=\frac{4}{5}t^4+ct^3-dt^{-1}$.

(g) $y''(t)+4y=\sec(2t)$
We will use the variation of parameter formula. We have $y_1(t)=\sin(2t)$, $y_2(t)=\cos(2t)$,

$W(y_1,y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = \sin(2t)\cdot(-2\sin(2t)) - \cos(2t)\cdot(2\cos(2t)) = -2$,
$\int \frac{y_2 g(t)}{W(y_1,y_2)(t)}dt = \int \frac{\cos(2t)\sec(2t)}{-2}dt = \int \frac{\cos(2t)}{-2\cos(2t)}dt = \int \frac{-1}{2}dt = \frac{-t}{2} + c$ and
$\int \frac{y_1 g(t)}{W(y_1,y_2)(t)}dt = \int \frac{\sin(2t)\sec(2t)}{-2}dt = \int \frac{\sin(2t)}{-2\cos(2t)}dt = \frac{\ln|\cos(2t)|}{4} + d$. We have used substitution $u = \cos(2t)$ and $du = -2\sin(2t)dt$.
Thus $y(t) = -\sin(2t)\cdot(\frac{-t}{2} + c) + \cos(2t)(\frac{\ln|\cos(2t)|}{4} + d) = -c\sin(2t) + d\cos(2t) + \frac{t\sin(2t)}{2} + \frac{\cos(2t)\ln|\cos(2t)|}{4}$.

(h) $y''(t) + 4y = \tan(2t)$
We will use the variation of parameter formula again. From previous example, we have $y_1(t) = \sin(2t)$, $y_2(t) = \cos(2t)$ and $W(y_1,y_2)(t) = -2$.
$\int \frac{y_2 g(t)}{W(y_1,y_2)(t)}dt = \int \frac{\cos(2t)\tan(2t)}{-2}dt = \int \frac{\cos(2t)\sin(2t)}{-2\cos(2t)}dt = \int \frac{-\sin(2t)}{2}dt = \frac{\cos(2t)}{4} + c$.
$\int \frac{y_1 g(t)}{W(y_1,y_2)(t)}dt = \int \frac{\sin(2t)\tan(2t)}{-2}dt = \int \frac{\sin(2t)sin(2t)}{-2\cos(2t)}dt == \int \frac{1-\cos^2(2t)}{-2\cos(2t)}dt = \int(\frac{\cos(2t)}{2} - \frac{\sec(2t)}{2})dt = \frac{\sin(2t)}{4} - \frac{\ln|\sec(2t)+\tan(2t)|}{4} + d$.
Thus $y(t) = -\sin(2t)\cdot(\frac{\cos(2t)}{4} + c) - \cos(2t)(\frac{\sin(2t)}{4} + \frac{\ln|\sec(2t)+\tan(2t)|}{4} + d) = -\cos(2t)\frac{\ln|\sec(2t)+\tan(2t)|}{4} - c\sin(2t) + d\cos(2t)$.

(i) $t^2y''(t) - t(t+2)y'(t) + (t+2)y(t) = t^4e^t(1+t)$.
Given that $y_1(t) = te^t$ is a solution of $t^2y''(t) - t(t+2)y'(t) + (t+2)y(t) = 0$.
This equation of this problem should be $t^2y''(t) - t(t+2)y'(t) + 2ty(t) = t^4e^t(1+t)$
Rewrite $t^2y''(t) - t(t+2)y'(t) + 2ty(t) = t^4e^t(1+t)$ as
$y''(t) - \frac{(t+2)}{t}y'(t) + \frac{2t}{t^2}y(t) = t^2e^t(1+t)$.
First, we find the solution of $y''(t) - \frac{(t+2)}{t}y'(t) + \frac{2t}{t^2}y(t) = 0$.
Let $p(t) = -\frac{t+2}{t}$. Let y_2 be another solution of $t^2y''(t) - t(t+2)y'(t) + (t+2)y(t) = 0$. We have $(\frac{y_2}{y_1})' = \frac{y_1y_2' - y_1'y_2}{y_1^2} = \frac{W(t)}{y_1^2} = \frac{Ce^{-\int p(t)dt}}{(te^t)^2} = \frac{Ce^{\int \frac{t+2}{t}dt}}{t^2e^{2t}} = \frac{Ce^{\int(1+\frac{2}{t})dt}}{t^2e^{2t}} = \frac{Ce^{(t+2\ln(t))}}{t^2e^{2t}} = \frac{Ce^te^{2\ln t}}{t^2e^{2t}} = \frac{Ce^tt^2}{t^2e^{2t}} = Ce^{-t}$. So $\frac{y_2}{y_1} = \int Ce^{-t}dt = -Ce^{-t} + D$ and $y_2 = y_1(-Ce^{-t} + D) = te^t(-Ce^{-t} + D) = -Ct + Dte^t$. So the general solution is $y = -Ct + Dte^t$. We may choose the second independent solution to be $y_2 = t$.
Now $y_1 = te^t$, $y_2 = t$,$y_1' = e^t + te^t$,$y_2' = 1$ and $W(y_1,y_2)(t) = y_1y_2' - y_2y_1' = te^t\cdot 1 - t(e^t + te^t) = -t^2e^t$. Recall that $g(t) = t^2e^t(1+t)$.
$\int \frac{y_2 g(t)}{W(y_1,y_2)(t)}dt = \int \frac{t\cdot t^2e^t(1+t)}{-t^2e^t}dt = \int(-t - t^2)dt = -\frac{t^2}{2} - \frac{t^3}{3} + c$.
$\int \frac{y_1 g(t)}{W(y_1,y_2)(t)}dt = \int \frac{te^t\cdot t^2e^t(1+t)}{-t^2e^t}dt = \int(-te^t - t^2e^t)dt = te^t - e^t - t^2e^t + d$.
Thus $y(t) = -te^t\cdot(-\frac{t^2}{2} - \frac{t^3}{3} + c) + t(te^t - e^t - t^2e^t + d) = (\frac{1}{3}t^4 - \frac{1}{2}t^3 + t^2 - t)e^t - cte^t + dt$.

(j) $(1-t)y''(t) + ty'(t) - y(t) = 2(t-1)^2e^{-t}$.
Given that $y_1(t) = t$ is a solution of $(1-t)y''(t) + ty'(t) - y(t) = 0$.

Rewrite $(1-t)y''(t)+ty'(t)-y(t)=2(t-1)^2e^{-t}$ as
$y''(t)+\frac{t}{1-t}y'(t)-\frac{1}{1-t}y(t)=\frac{2(t-1)^2e^{-t}}{1-t}=2(1-t)e^{-t}$.
First, we find the solution of $y''(t)+\frac{t}{1-t}y'(t)-\frac{1}{1-t}y(t)=0$.
Let $p(t)=\frac{t}{1-t}$. Let y_2 be another solution of $y''(t)+\frac{t}{1-t}y'(t)-\frac{1}{1-t}y(t)=0$. We have $(\frac{y_2}{y_1})'=\frac{y_1y_2'-y_1'y_2}{y_1^2}=\frac{W(t)}{y_1^2}=\frac{Ce^{-\int p(t)dt}}{t^2}=\frac{Ce^{\int\frac{-t}{1-t}dt}}{t^2}=\frac{Ce^{\int\frac{-t+1-1}{1-t}dt}}{t^2}=\frac{Ce^{\int(1+\frac{1}{t-1})dt}}{t^2e^{2t}}=\frac{Ce^{(t+\ln(t-1))}}{t^2}=\frac{Ce^{-t}(t-1)}{t^2}=\frac{Ce^{t}(t-1)}{t^2}=Ce^t(\frac{1}{t}-\frac{1}{t^2})$. So $\frac{y_2}{y_1}=\int Ce^t(\frac{1}{t}-\frac{1}{t^2})dt=C\frac{e^t}{t}+D$ and $y_2=y_1(C\frac{e^t}{t}+D)=t(C\frac{e^t}{t}+D)=Ce^t+Dt$. So the general solution is $y=Ce^t+Dt$. We may choose the second independent solution to be $y_2=e^t$.
Now $y_1=t$, $y_2=e^t$,$y_1'=1$,$y_2'=e^t$ and $W(y_1,y_2)(t)=y_1y_2'-y_2y_1'=te^t-e^t=e^t(t-1)$. Recall that $g(t)=2(1-t)e^{-t}$.
$\int\frac{y_2g(t)}{W(y_1,y_2)(t)}dt=\int\frac{e^t\cdot 2(1-t)e^{-t}}{e^t(t-1)}dt=\int(-2e^{-t})dt=2e^{-t}+c$.
$\int\frac{y_1g(t)}{W(y_1,y_2)(t)}dt=\int\frac{t\cdot 2(1-t)e^{-t}}{e^t(t-1)}dt=\int(-2te^{-2t})dt=\frac{1}{2}e^{-2t}(2t+1)+d$.
Thus $y(t)=-t\cdot(2e^{-t}+c)+e^t(\frac{1}{2}e^{-2t}(2t+1)+d)=-te^{-t}+\frac{1}{2}e^{-t}-ct+de^t$.

(5) Find the solution of the following initial value problems.

(a) $y''(t)+4y=2\sin(2t)+3\cos(t)$, $y(0)=3$ and $y'(0)=5$. You may use the result in 4b.
From (4b), we have $y(t)=-\frac{1}{2}t\cos(2t)+\cos(t)+c_1\sin(2t)+c_2\cos(2t)$. So $y'(t)=-\frac{1}{2}\cos(2t)+t\sin(2t)-\sin(t)+2c_1\cos(2t)-2c_2\sin(2t)$. Using $y(0)=3$ and $y'(0)=5$, we have $1+c_2=3$ and $-\frac{1}{2}+2c_1=5$. Thus $c_1=\frac{11}{4}$, $c_2=2$ and $y(t)=-\frac{1}{2}t\cos(2t)+\cos(t)+\frac{11}{4}\sin(2t)+2\cos(2t)$.

(b) $y''(t)-ty'(t)+\sin(y(t))=0$, $y(1)=0$ and $y'(1)=0$.
One can check that $y(t)=0$ is a solution of $y''(t)-ty'(t)+\sin(y(t))=0$. By the uniqueness and existence Theorem, we have $y(t)=0$.

(6) What is the form of the particular solution of the following equations? (You don't have to find the particular solution. For example, the form of the particular solution of $y''+y=\sin(t)$ is $y_p(t)=ct\sin(t)+dt\sin(t)$.)

(a) $y''(t)+4y=t^2e^{-4t}$ Solving $r^2+4=0$, we have $r=\pm 2i$. The solution of $y''(t)+4y=0$ is $y(t)=c_1\sin(2t)+c_2\cos(2t)$.
So the particular solution of $y''(t)+4y=t^2e^{-4t}$ is $y_p(t)=ct^2e^{-4t}+dte^{-4t}+fe^{-4t}$.

(b) $y''(t)+4y+4y(t)=te^{-2t}+e^{2t}$
Solving $r^2+4r+4=0$, we have $r=-2$. The solution of $y''(t)+4y+4y(t)=0$ is $y(t)=c_1e^{-2t}+c_2te^{-2t}$.

Note that te^{-2t} is a solution of $y''(t)+4y+4y(t)=0$ and e^{2t} is not a solution of $y''(t)+4y+4y(t)=0$. So the particular solution of $y''(t)+4y+4y(t)=te^{-2t}+e^{2t}$ is $y_p(t)=ct^2e^{-2t}+de^{2t}$.

(c) $y''(t)+2y'(t)+2y(t)=2te^t\cos(t)$. Solving $r^2+2r+2=0$, we have $r=-1\pm i$. The solution of $y''(t)+2y'(t)+2y(t)=0$ is $y(t)=c_1e^t\cos(t)+c_2e^t\cos(t)$. Note that $e^t\cos(t)$ is a solution of $y''(t)+2y'(t)+2y(t)=0$ So the particular solution of $y''(t)+2y'(t)+2y(t)=2te^t\cos(t)$ is $y_p(t)=ct^2e^t\cos(t)+dt^2e^t\sin(t)+fte^t\cos(t)+gte^t\sin(t)$.

(d) $y''(t)+2y'(t)+2y(t)=2te^t\sin(2t)$. Solving $r^2+2r+2=0$, we have $r=-1\pm i$. The solution of $y''(t)+2y'(t)+2y(t)=0$ is $y(t)=c_1e^t\cos(t)+c_2e^t\cos(t)$. Note that $e^t\sin(2t)$ is not a solution of $y''(t)+2y'(t)+2y(t)=0$ So the particular solution of $y''(t)+2y'(t)+2y(t)=2te^t\sin(2t)$ is $y_p(t)=cte^t\sin(2t)+dte^t\cos(2t)+fe^t\sin(2t)+ge^t\cos(2t)$.

(e) $y''(t)+4y=t\sin(2t)+3t\cos(2t)$. Solving $r^2+4=0$, we have $r=\pm 2i$. The solution of $y''(t)+4y=0$ is $y(t)=c_1\sin(2t)+c_2\cos(2t)$. Note that $\sin(2t)$ and $\cos(2t)$ are solutions of $y''(t)+4y=0$. So the particular solution of $y''(t)+4y(t)=t\sin(2t)+3t\cos(2t)$ is $y_p(t)=ct^2\sin(2t)+dt^2\cos(2t)+ft\sin(2t)+gt\cos(2t)$.

(7) Express the solution of the following equation in the form of $y=Ae^{Bt}\cos(Ct-D)$.

(a) $y''(t)+2y'(t)+2y(t)=0$, $y(0)=2$ and $y'(0)=3$. Solving $r^2+2r+2=0$, we have $r=-1\pm i$. So the general solution of $y''(t)+2y'(t)+2y(t)=0$ is $y(t)=c_1e^{-t}\sin(t)+c_2e^{-t}\cos(t)$. We have $y'(t)=-c_1e^{-t}\sin(t)+c_1e^{-t}\cos(t)-c_2e^{-t}\cos(t)-c_2e^{-t}\sin(t)=(-c_1-c_2)e^{-t}\sin(t)+(c_1-c_2)e^{-t}\cos(t)$. Using $y(0)=2$ and $y'(0)=3$, we have $c_2=2$ and $c_1-c_2=3$. This gives $c_1=c_2+3=5$ and $c_2=2$. Thus $y(t)=5e^{-t}\sin(t)+2e^{-t}\cos(t)=e^{-t}(5\sin(t)+2\cos(t))=e^{-t}\sqrt{29}(\frac{5}{\sqrt{29}}\sin(t)+\frac{2}{\sqrt{29}}\cos(t))=\sqrt{29}e^{-t}\cos(t-\theta)$ where θ is determined by $\cos(\theta)=\frac{2}{\sqrt{29}}$ and $\sin(\theta)=\frac{5}{\sqrt{29}}$.

(b) $y''(t)+4y'(t)+5y(t)=0$, $y(0)=2$ and $y'(0)=3$. Solving $r^2+4r+5=0$, we have $r=-2\pm i$. So the general solution of $y''(t)+4y'(t)+5y(t)=0$ is $y(t)=c_1e^{-2t}\sin(t)+c_2e^{-2t}\cos(t)$. We have $y'(t)=-2c_1e^{-2t}\sin(t)+c_1e^{-2t}\cos(t)-2c_2e^{-2t}\cos(t)-c_2e^{-2t}\sin(t)=(-2c_1-c_2)e^{-2t}\sin(t)+(c_1-2c_2)e^{-2t}\cos(t)$. Using $y(0)=2$ and $y'(0)=3$, we have $c_2=2$ and $c_1-2c_2=3$. This gives $c_1=7$ and $c_2=2$. Thus $y(t)=7e^{-2t}\sin(t)+2e^{-2t}\cos(t)=e^{-2t}(7\sin(t)+2\cos(t))=e^{-2t}\sqrt{53}(\frac{7}{\sqrt{53}}\sin(t)+\frac{2}{\sqrt{53}}\cos(t))=\sqrt{53}e^{-2t}\cos(t-\theta)$ where θ is determined by $\cos(\theta)=\frac{2}{\sqrt{53}}$ and $\sin(\theta)=\frac{7}{\sqrt{53}}$.

(8) Solve the following problems and describe the behavior of the solutions.

(a) $y''(t) + 4y(t) = A\cos(wt)$ if $w \neq 2$.
The solution of $y''(t) + 4y(t) = 0$ is $y(t) = c_1\cos(2t) + c_2\sin(2t)$ If $w \neq 2$, we can try $y_p(t) = c\sin(wt) + d\cos(wt)$. Then $y_p' = cw\cos(wt) - dw\sin(wt)$ and $y_p'' = -cw^2\sin(wt) - dw^2\cos(wt)$. So $y_p'' + 4y_p = -cw^2\sin(wt) - dw^2\cos(wt) + 4(c\sin(wt) + d\cos(wt)) = c(4-w^2)\sin(wt) + d(4-w^2)\cos(wt) = A\cos(wt)$ if $d(4-w^2) = A$ and $c(4-w^2) = 0$. Using $w \neq 2$, we have $c = 0$ and $d = \frac{A}{4-w^2}$. Thus $y(t) = \frac{A}{4-w^2}\cos(wt) + c_1\cos(2t) + c_2\sin(2t)$.

(b) $y''(t) + 4y(t) = A\cos(2t)$.
The solution of $y''(t)+4y(t) = 0$ is $y(t) = c_1\cos(2t)+c_2\sin(2t)$ We can try $y_p(t) = ct\sin(2t) + dt\cos(2t)$. Then $y_p' = c\sin(2t) + 2ct\cos(2t) + d\cos(2t) - 2dt\sin(2t)$, $y_p'' = 2c\cos(2t)+2c\cos(2t) - 4ct\sin(2t) - 2d\sin(2t) - 2d\sin(2t) - 4dt\cos(2t)$ and $y_p''+4y_p = 2c\cos(2t)+2c\cos(2t)-4ct\sin(2t)-2d\sin(2t)-2d\sin(2t)-4dt\cos(2t)+4ct\sin(2t) + 4dt\cos(2t) = 4c\cos(2t) - 4d\sin(2t)$. Thus $y_p'' + 4y_p = A\cos(2t)$ if $4c = A$ and $d = 0$. Hence $y(t) = \frac{A}{4}t\sin(2t) + c_1\cos(2t) + c_2\sin(2t)$.