Solutions to Review Problems for Midterm III

(1) Find the general solution of the following differential equations.

(a) $y^{(6)}(t) + 64y(t) = 0$.

The characteristic equation of $y^{(6)}(t) + 64y(t) = 0$ is $r^6 + 64 = 0$. Note that $-64 = 64e^{i(\pi+2k\pi)}$ where k is an integer. Solving $r^6 + 64 = 0$ is the same as solving $r^6 = -64 = 64e^{i(\pi+2k\pi)}$. Therefore $r = \sqrt[6]{64}e^{i\frac{(\pi+2k\pi)}{6}} = 2e^{i\frac{(\pi+2k\pi)}{6}} = 2(\cos(\frac{(\pi+2k\pi)}{6}) + i\sin(\frac{(\pi+2k\pi)}{6}))$ where $k = 0, 1, 2, \cdots, 5$.

Let $r_k = 2(\cos(\frac{(\pi+2k\pi)}{6}) + i\sin(\frac{(\pi+2k\pi)}{6}))$ where $k = 0, 1, 2, \cdots, 5$. Therefore $r_0 = 2(\cos(\frac{\pi}{6})+i\sin(\frac{\pi}{6})) = \sqrt{3}+i$, $r_1 = 2(\cos(\frac{\pi}{2})+i\sin(\frac{\pi}{2})) = 2i$, $r_2 = 2(\cos(\frac{5\pi}{6})+i\sin(\frac{5\pi}{6})) = -\sqrt{3}+i$, $r_3 = 2(\cos(\frac{7\pi}{6}) + i\sin(\frac{7\pi}{6})) = -\sqrt{3} - i$, $r_4 = 2(\cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2})) = -2i$ and $r_5 = 2(\cos(\frac{11\pi}{6}) + i\sin(\frac{11\pi}{6})) = \sqrt{3} - i$. So the roots of characteristic equations are $\sqrt{3} \pm i$, $\pm 2i$ and $-\sqrt{3} \pm i$.

The general solution is $y(t) = c_1e^{\sqrt{3}t}\cos(t)+c_2e^{\sqrt{3}t}\sin(t)+c_3\cos(2t)+c_4\sin(2t)+c_5e^{-\sqrt{3}t}\cos(t) + c_6e^{-\sqrt{3}t}\sin(t)$.

(b) $y^{(3)}(t) + 3y^{(2)}(t) + 2y'(t) = 0$.

The characteristic equation of $y^{(3)}(t) + 3y^{(2)}(t) + 2y'(t) = 0$ is $r^3 + 3r^2 + 2r = r(r^2 + 3r + 2) = r(r + 1)(r + 2) = 0$. Its roots are $r = 0$, $r = -1$ and $r = -2$. The general solution is $y(t) = c_1 + c_2e^{-t} + c_3e^{-2t}$.

(c) $y^{(4)}(t) - 8y^{(2)}(t) + 16y = 0$.

The characteristic equation of $y^{(4)}(t) - 8y^{(2)}(t) + 16y = 0$ is $r^4 - 8r^2 + 16 = (r^2 - 4)^2 = (r - 2)^2(r + 2)^2$. Its roots are $r = 2$ with multiplicity 2 and $r = -2$ with multiplicity 2. The general solution is $y(t) = c_1e^{2t}+c_2te^{2t}+c_3e^{-2t}+c_4te^{-2t}$.

Note: Compare with the problems $y^{(4)}(t)+8y^{(2)}(t)+16y = 0$. The characteristic equation of $y^{(4)}(t) + 8y^{(2)}(t) + 16y = 0$ is $r^4 + 8r^2 + 16 = (r^2 + 4)^2 = 0$. Its roots are $r = \pm 2i$ with multiplicity 2. The general solution is $y(t) = c_1\cos(2t) + c_2\sin(2t) + c_3t\cos(2t) + c_4t\sin(2t)$.

(d) $y^{(6)}(t) + 2y^{(3)}(t) + y(t) = 0$.

The characteristic equation of $y^{(6)}(t) + 2y^{(3)}(t) + y(t) = 0$ is $r^6 + 2r^3 + 1 = 0$. Note that $r^6 + 2r^3 + 1 = (r^3 + 1)^2$ and $-1 = 1e^{i(\pi+2k\pi)}$ where k is an integer. Solving $r^3 + 1 = 0$ is the same as solving $r^3 = -1 = e^{i(\pi+2k\pi)}$.

Therefore $r = e^{i\frac{(\pi+2k\pi)}{3}} = e^{i\frac{(\pi+2k\pi)}{3}} = \cos(\frac{(\pi+2k\pi)}{3})+i\sin(\frac{(\pi+2k\pi)}{3})$ where $k = 0, 1, 2$.

Let $r_k = \cos(\frac{(\pi+2k\pi)}{3}) + i\sin(\frac{(\pi+2k\pi)}{3})$ where $k = 0, 1, 2$.

Therefore $r_0 = \cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3}) = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $r_1 = (\cos(\pi) + i\sin(\pi) = -1$, $r_2 = 2(\cos(\frac{5\pi}{3}) + i\sin(\frac{5\pi}{3})) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$, Note that $r_0 = \overline{r_2}$. Thus the root of $(r^3 + 1)^2$ is $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ with multiplicity 2 and $r_1 = -1$ with multiplicity 2.

The general solution is $y(t) = c_1e^{\frac{t}{2}}\cos(\frac{\sqrt{3}}{2}t) + c_2e^{\frac{t}{2}}\sin(\frac{\sqrt{3}}{2}t) + c_3te^{\frac{t}{2}}\cos(\frac{\sqrt{3}}{2}t) + c_4te^{\frac{t}{2}}\sin(\frac{\sqrt{3}}{2}t) + c_5e^{-t} + c_6te^{-t}$.

(e) $(D^2-4D+13)^2(D-2)^2y(t)=0$.
The characteristic equation of $(D^2-4D+13)^2(D-2)^2y(t)=0$ is $(r^2-4r+13)^2(r-2)^2=0$. Its roots are $r=2\pm 3i$ with multiplicity 2 and $r=2$ with multiplicity 2 . The general solution is $y(t)=c_1e^{2t}\cos(3t)+c_2e^{2t}\sin(3t)+c_3te^{2t}\cos(3t)+c_4e^{2t}\sin(3t)+c_5e^{2t}+c_6te^{2t}$.

(2) Use the method of Annihilators to find the form of particular solution of the following problems.

(a) $(D^3-2D^2+D)y=t+\cos(t)+t\sin(t)+t^2e^t$.
Solving the characteristic equation $r^3-2r^2+2r=r(r^2-2r+r)=r(r-1)^2=0$, we have $r=0,\,1,\,1$. The solutions of $(D^3-2D^2+D)y=0$ are spanned by 1, e^t and te^t.
Now we should find the annihilator of $t+\cos(t)+t\sin(t)+t^2e^t$. We have $D^2(t)=0$, $(D^2+1)^2(\cos(t)+t\sin(t))=0$ and $(D-1)^3(t^2e^t)=0$.
So $D^2(D^2+1)^2(D-1)^3(t+\cos(t)+t\sin(t)+t^2e^t)=0$.
The given equation is $(D^3-2D^2+D)y=t+\cos(t)+t\sin(t)+t^2e^t$. Applying the annihilator $D^2(D^2+1)^2(D-1)^3$ to the equation above, we have

$$D^2(D^2+1)^2(D-1)^3(D^3-2D^2+D)y$$

$$=D^2(D^2+1)^2(D-1)^3(t+\cos(t)+t\sin(t)+t^2e^t)=0.$$

Solving the characteristic equation
$r^2(r^2+1)^2(r-1)^3(r^3-2r^2+r)$
$=r^2(r^2+1)^2(r-1)^3r(r-1)^2=r^3(r^2+1)^2(r-1)^5=0$, we have $r=0$ with multiplicity 3, $\pm i$ with multiplicity 2, and 1 with multiplicity 5. The solution of $(D^3-2D^2+D)y=t+\cos(t)+t\sin(t)+t^2e^t$ is spanned by 1, t, t^2, $\cos(t)$, $\sin(t)$, $t\cos(t)$, $t\sin(t)$, e^t, te^t, t^2e^t, t^3e^t and t^4e^t. Excluding those functions (1, e^t and te^t) appeared as the solution of $(D^3+2D^2+D)y=0$, we know that the particular solution is of the form
$y_p(t)=c_1t+c_2t^2+c_3\cos(t)+c_4\sin(t)+c_5t\cos(t)+c_6t\sin(t)+c_7t^2e^t+c_8t^3e^t+c_9t^4e^t$.

(b) $(D^3+D)y=t+\cos(t)+t\sin(t)+t^2e^t$.
Solving the characteristic equation $r^3+r=r(r^2+1)=0$, we have $r=0,\,\pm i$. The solutions of $(D^3+D)y=0$ are spanned by 1, $\cos(t)$ and $\sin(t)$.
From previous question, we know that $D^2(D^2+1)^2(D-1)^3(t+\cos(t)+t\sin(t)+t^2e^t)=0$. So $D^2(D^2+1)^2(D-1)^3(D^3+D)y=0$. Solving the characteristic equation
$r^2(r^2+1)^2(r-1)^3(r^3+r)$
$=r^2(r^2+1)^2(r-1)^3r(r^2+1)=r^3(r^2+1)^3(r-1)^2=0$, we have $r=0$ with multiplicity 3, $\pm i$ with multiplicity 3, and 1 with multiplicity 2. The solution of

$(D^3+D)y = t+\cos(t)+t\sin(t)+t^2e^t$ is spanned by 1, t, t^2, $\cos(t)$, $\sin(t)$, $t\cos(t)$, $t\sin(t)$, $t^2\cos(t)$, $t^2\sin(t)$,e^t, and te^t. Excluding those functions (1, $\cos(t)$ and $\sin(t)$) appeared as the solution of $(D^3+D)y = 0$, we know that the particular solution is of the form $y_p(t) = c_0t + c_1t^2 + c_2t\cos(t) + c_3t\sin(t) + c_4t^2\cos(t) + c_5t^2\sin(t) + c_5e^t + c_6te^t + c_7t^2e^t$.

(c) $y''(t) + 2y'(t) + 2y(t) = 3te^{-t}\cos(t)$.
Solving the characteristic equation $r^2 + 2r + 2 = 0$, we have $r = -1 \pm i$. The solutions of $(D^2 + 2D + 2)y = 0$ are spanned by $e^{-t}\cos(t)$ and $e^{-t}\sin(t)$.
The annihilator of $te^{-t}\cos(t)$ is $(D^2+2D+2)^2$. We have $(D^2+2D+2)^2(te^{-t}\cos(t)) = 0$. From $(D^2+2D+2)(y) = 3te^{-t}\cos(t)$, we get $(D^2+2D+2)^2(D^2+2D+2)(y) = (D^2+2D+2)^2(3te^{-t}\cos(t)) = 0$ and $(D^2+2D+2)^3(y) = 0$. Solving the characteristic equation $(r^2 + 2r + 2)^3 = 0$, we have $r = -1 \pm i$ with multiplicity 3. The solutions of $(D^2 + 2D + 2)^3y = 0$ are spanned by $e^{-t}\cos(t)$, $e^{-t}\sin(t)$, $te^{-t}\cos(t)$, $te^{-t}\sin(t)$, $t^2e^{-t}\cos(t)$ and $t^2e^{-t}\sin(t)$. Excluding those functions ($e^{-t}\cos(t)$ and $e^{-t}\sin(t)$) appeared as the solution of $(D^2+2D+2)y = 0$, we know that the particular solution is of the form $y_p(t) = c_1te^{-t}\cos(t) + c_2te^{-t}\sin(t) + c_3t^2e^{-t}\cos(t) + c_4t^2e^{-t}\sin(t)$.

(3) Use Laplace's transform to find the solution of the following initial value problems.

(a) $y^{(3)}(t) - 3y^{(2)}(t) + 2y'(t) = e^{4t}$ with $y(0) = 1$, $y'(0) = 0$ and $y''(0) = 0$.
Taking the Laplace's transform of the equation, we have
$L(y^{(3)}(t) - 3y^{(2)}(t) + 2y'(t)) = L(e^{4t})$
$\Rightarrow s^3L(y) - s^2 - 3s^2L(y) + 3s + 2sL(y) - 2 = \frac{1}{s-4}$
$\Rightarrow (s^3 - 3s^2 + 2s)L(y) = s^2 - 3s + 2 + \frac{1}{s-4}$
$\Rightarrow L(y) = \frac{s^2-3s+2}{(s^3-3s^2+2s)} + \frac{1}{(s-4)(s^3-3s^2+2s)}$
$\Rightarrow L(y) = \frac{1}{s} + \frac{1}{(s-4)(s^3-3s^2+2s)}$

Using partial fraction, we have
$\frac{1}{(s-4)(s^3-3s^2+2s)} = \frac{1}{s(s-1)(s-2)(s-4)} = \frac{a}{s} + \frac{b}{(s-1)} + \frac{c}{(s-2)} + d\frac{c}{(s-4)}$.
Multiplying $s(s-1)(s-2)(s-4)$, we get
$1 = a(s-1)(s-2)(s-4) + bs(s-2)(s-4) + cs(s-1)(s-4) + ds(s-1)(s-2)$.
Plugging $s = 0$, we get $a = -\frac{1}{8}$. Plugging $s = 1$, we get $b = \frac{1}{3}$. Plugging $s = 2$, we get $c = -\frac{1}{4}$. Plugging $s = 4$, we get $d = \frac{1}{24}$. So we have
$\frac{1}{(s-4)(s^3-3s^2+2s)} = -\frac{1}{8}\frac{1}{s} + \frac{1}{3}\frac{1}{(s-1)} - \frac{1}{4}\frac{1}{(s-2)} + \frac{1}{24}\frac{1}{(s-4)}$.
So we have $L(y) = \frac{1}{s} + \frac{1}{(s-4)(s^3-3s^2+2s)}$
$= \frac{1}{s} - \frac{1}{8}\frac{1}{s} + \frac{1}{3}\frac{1}{(s-1)} - \frac{1}{4}\frac{1}{(s-2)} + \frac{1}{24}\frac{1}{(s-4)}$

$= \frac{7}{8}\frac{1}{s} + \frac{1}{3}\frac{1}{(s-1)} - \frac{1}{4}\frac{1}{(s-2)} + \frac{1}{24}\frac{1}{(s-4)}$
and $y(t) = L^{-1}(\frac{7}{8}\frac{1}{s} + \frac{1}{3}\frac{1}{(s-1)} - \frac{1}{4}\frac{1}{(s-2)} + \frac{1}{24}\frac{1}{(s-4)}) = \frac{7}{8} + \frac{1}{3}e^t - \frac{1}{4}e^{2t} + \frac{1}{24}e^{4t}$

(b) $y''(t) + y(t) = \sin(2t)$ with $y(0) = 0$, $y'(0) = 0$.
Taking the Laplace's transform and using the conditions, we have
$L(y''(t) + y(t)) = L(\sin(2t))$
$\Rightarrow (s^2+1)Y(s) = \frac{2}{s^2+4}$
$\Rightarrow Y(s) = \frac{2}{(s^2+4)(s^2+1)}$
Using partial fraction, we have $\frac{2}{(s^2+4)(s^2+1)} = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4}$. Multiplying $(s^2+4)(s^2+1)$, we get $2 = (as+b)(s^2+4) + (cs+d)(s^2+1)$ and
$2 = as^3+bs^2+4as+4b+cs^3+ds^2+cs+d = (a+c)s^3+(b+d)s^2+(4a+c)s+4b+d$.
Comparing the coefficient, we get $a+c = 0$, $b+d = 0$, $4a+c = 0$ and $4b+d = 2$. From $a+c = 0$ and $4a+c = 0$, we get $a = 0$ and $c = 0$. From $b+d = 0$ and $4b+d = 2$, we get $b = \frac{2}{3}$ and $d = -\frac{2}{3}$. So $Y(s) = \frac{2}{(s^2+4)(s^2+1)} = \frac{2}{3}\frac{1}{s^2+1} - \frac{2}{3}\frac{1}{s^2+4}$.
Hence $y(t) = L^{-1}(\frac{2}{3}\frac{1}{s^2+1} - \frac{2}{3}\frac{1}{s^2+4}) = \frac{2}{3}\sin(t) - \frac{1}{3}\sin(2t)$. Note that $L^{-1}(\frac{1}{s^2+a^2}) = \frac{1}{a}\sin(at)$.

(c) $y''(t) + 4y = g(t)$ with $y(0) = 0$ and $y'(0) = 0$ where

$$g(t) = \begin{cases} 0, & 0 \leq t < 2, \\ 3(t-2), & 2 \leq t < 4, \\ 6, & 4 \leq t. \end{cases}$$

We have $g(t) = 3(t-2)u_{2,4}(t) + 6u_4(t) = 3(t-2)(u_2(t) - u_4(t)) + 6u_4(t) = 3(t-2)u_2(t) - (3t-12)u_4(t) = 3(t-2)u_2(t) - 3(t-4)u_4(t)$. Let $h(t-2) = t-2$ and $k(t-4) = t-4$. Then $h(t) = t$ and $k(t) = t$. So $g(t) = 3h(t-2)u_2(t) - 3k(t-4)u_4(t)$ and $L(g(t)) = L(3h(t-2)u_2(t) - 3k(t-4)u_4(t)) = 3e^{-2s}L(h(t)) - 3e^{-4s}L(k(t)) = 3\frac{e^{-2s}}{s^2} - 3\frac{e^{-4s}}{s^2}$.
$L(y''(t) + 4y(t)) = L(g(t)) = 3\frac{e^{-2s}}{s^2} - 3\frac{e^{-4s}}{s^2}$
$\Rightarrow (s2+4)Y(s) = 3\frac{e^{-2s}}{s^2} - 3\frac{e^{-4s}}{s^2}$
$\Rightarrow Y(s) = 3\frac{e^{-2s}}{s^2(s^2+4)} - 3\frac{e^{-4s}}{s^2(s^2+4)}$
$\Rightarrow Y(s) = 3\frac{e^{-2s}}{s^2(s^2+4))} - 3\frac{e^{-4s}}{s^2(s^2+4)}$
Using partial fraction, we have $\frac{1}{s^2(s^2+4)} = \frac{a}{s} + \frac{b}{s^2} + \frac{cs+d}{(s^2+4)}$. This implies that $1 = as(s^2+4) + b(s^2+4) + cs^3 + ds^2 = (a+c)s^3 + (b+d)s^2 + 4as + 4b$. Comparing the coefficient, we get $a+c = 0$, $b+d = 0$, $4a = 0$ and $4b = 1$. We get $a = 0$ and $c = 0$, $b = \frac{1}{4}$ and $d = -\frac{1}{4}$. So $\frac{1}{s^2(s^2+4)} = \frac{1}{4}\frac{1}{s^2} - \frac{1}{4}\frac{1}{(s^2+4)}$.
$Y(s) = 3\frac{e^{-2s}}{s^2(s^2+4)} - 3\frac{e^{-4s}}{s^2(s^2+4)} = \frac{3}{4}e^{-2s}(\frac{1}{s^2} - \frac{1}{(s^2+4)}) - \frac{3}{4}e^{-4s}(\frac{1}{s^2} - \frac{1}{(s^2+4)})$.
Let $f(t) = L^{-1}(\frac{1}{s^2} - \frac{1}{(s^2+4)}) = t - \frac{1}{2}\sin(2t)$. Hence $y(t) = L^{-1}(\frac{3}{4}e^{-2s}(\frac{1}{s^2} - \frac{1}{(s^2+4)}) - \frac{3}{4}e^{-4s}(\frac{1}{s^2} - \frac{1}{(s^2+4)})) = \frac{3}{4}u_2(t)f(t-2) - \frac{3}{4}u_4(t)f(t-4)$.

(d) $y''(t) + 4y'(t) + 4y = g(t)$ with $y(0) = 0$ and $y'(0) = 0$ where

$$g(t) = \begin{cases} 0, & 0 \leq t < 2, \\ 3(t-2), & 2 \leq t < 4, \\ 6, & 4 \leq t. \end{cases}$$

From previous question, we have $L(g(t)) = 3\frac{e^{-2s}}{s^2} - 3\frac{e^{-4s}}{s^2}$.
$L(y''(t) + 4y'(t) + 4y(t)) = L(g(t)) = 3\frac{e^{-2s}}{s^2} - 3\frac{e^{-4s}}{s^2}$
$\Rightarrow (s2 + 4s + 4)Y(s) = 3\frac{e^{-2s}}{s^2} - 3\frac{e^{-4s}}{s^2}$
$\Rightarrow Y(s) = 3\frac{e^{-2s}}{s^2(s^2+4s+4)} - 3\frac{e^{-4s}}{s^2(s^2+4s+4)}$
$\Rightarrow Y(s) = 3\frac{e^{-2s}}{s^2(s+2)^2} - 3\frac{e^{-4s}}{s^2(s+2)^2}$
Using partial fraction, we have $\frac{1}{s^2(s+2)^2} = \frac{a}{s} + \frac{b}{s^2} + \frac{c(s+2)+d}{(s+2)^2}$. This implies that $1 = as(s+2)^2 + b(s+2)^2 + cs^2(s+2) + ds^2$. Plugging in $s = 0$, we get $b = \frac{1}{4}$. Plugging in $s = -2$, we get $d = \frac{1}{4}$. Hence $1 = as(s+2)^2 + \frac{1}{4}(s+2)^2 + cs^2(s+2) + \frac{1}{4}s^2$. Plugging in $s = 1$ and $s = -1$, we have $1 = 9a + \frac{9}{4} + 3c + \frac{1}{4}$ and $1 = -a + \frac{1}{4} + c + \frac{1}{4}$. This gives $9a + 3c = -\frac{3}{2}$ and $-a + c = \frac{1}{2}$. Hence $a = -\frac{1}{4}$ and $c = \frac{1}{4}$. Now we have $\frac{1}{s^2(s+2)^2} = -\frac{1}{4}\frac{1}{s} + \frac{1}{4}\frac{1}{s^2} + \frac{1}{4}\frac{1}{(s+2)} + \frac{1}{4}\frac{1}{(s+2)^2}$. Let $f(t) = L^{-1}(-\frac{1}{4}\frac{1}{s} + \frac{1}{4}\frac{1}{s^2} + \frac{1}{4}\frac{1}{(s+2)} + \frac{1}{4}\frac{1}{(s+2)^2}) = -\frac{1}{4} + \frac{1}{4}t + \frac{1}{4}e^{-2}t + \frac{1}{4}te^{-2t}$. Then $y(t) = L^{-1}(3\frac{e^{-2s}}{s^2(s+2)^2} - 3\frac{e^{-4s}}{s^2(s+2)^2}) = 3u_2(t)f(t-2) - 3u_4(t)f(t-4)$.

(e) $y''(t) + 4y'(t) + 5y = g(t)$ with $y(0) = 0$ and $y'(0) = 0$ where

$$g(t) = \begin{cases} 0, & 0 \leq t < 2, \\ 1, & 2 \leq t < 4, \\ 0, & 4 \leq t. \end{cases}$$

We have $g(t) = u_{2,4}(t) = u_2(t) - u_4(t)$ and $L(g(t)) = e^{-2s} - e^{-4s}$. Taking the Laplace transform, we get $L(y''(t)+5y'(t)+5y) = L(g(t))$ and $(s^2+4s+5)Y(s) = e^{-2s} - e^{-4s}$.
$\Rightarrow Y(s) = \frac{e^{-2s}}{(s^2+4s+5)} - \frac{e^{-4s}}{(s^2+4s+5)}$.
Note that $\frac{1}{(s^2+4s+5)} = \frac{1}{(s+2)^2+1}$ and $f(t) = L^{-1}(\frac{1}{(s+2)^2+1}) = e^{-2t}\sin(t)$.
Then $y(t) = L^{-1}(\frac{e^{-2s}}{(s^2+4s+5)} - \frac{e^{-4s}}{(s^2+4s+5)}) = u_2(t)f(t-2) - u_4(t)f(t-4)$.

(f) $y''(t) + 5y'(t) + 4y(t) = \delta(t-2)$, with $y(0) = 0$ and $y'(0) = 0$.
Taking the Laplace transform $L(y''(t) + 5y'(t) + 4y(t)) = L(\delta(t-2))$, we have $(s^2 + 5s + 4)Y(s) = e^{-2s}$.
$\Rightarrow Y(s) = \frac{e^{-2s}}{(s^2+5s+4)} = \frac{e^{-2s}}{((s+1)(s+4))} = e^{-2s}(\frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s+4})$
$= \frac{1}{3}\frac{e^{-2s}}{s+1} - \frac{1}{3}\frac{e^{-2s}}{s+4}$.
Let $f(t) = L^{-1}(\frac{1}{s+1}) = e^{-t}$ and $g(t) = L^{-1}(\frac{1}{s+4}) = e^{-4t}$

We have $y(t) = L^{-1}(\frac{1}{3}\frac{e^{-2s}}{s+1} - \frac{1}{3}\frac{e^{-2s}}{s+4}) = \frac{1}{3}u_2(t)f(t-2) - \frac{1}{3}u_2(t)g(t-2) = \frac{1}{3}u_2(t)e^{-(t-2)} - \frac{1}{3}u_2(t)e^{-4(t-2)}$.

(g) $y''(t) + 4y'(t) + 5y(t) = \delta(t-2)$, with $y(0) = 1$ and $y'(0) = 1$.
Taking the Laplace transform $L(y''(t) + 4y'(t) + 5y(t)) = L(\delta(t-2))$, we have $(s^2 + 4s + 5)Y(s) - s - 5 = e^{-2s}$.
$\Rightarrow Y(s) = \frac{e^{-2s}(s+5)}{(s^2+4s+5)} + \frac{e^{-2s}}{(s^2+4s+5)} = \frac{e^{-2s}(s+5)}{(s+2)^2+1} + \frac{e^{-2s}}{(s+2)^2+1}$
$= \frac{e^{-2s}(s+2)}{(s+2)^2+1} + 4\frac{e^{-2s}}{(s+2)^2+1}$.
Let $f(t) = L^{-1}(\frac{(s+2)}{(s+2)^2+1}) = e^{-2t}\cos(t)$ and $g(t) = L^{-1}(\frac{1}{(s+2)^2+1}) = e^{-2t}\sin(t)$
We have $y(t) = L^{-1}(\frac{e^{-2s}(s+2)}{(s+2)^2+1} + 4\frac{e^{-2s}}{(s+2)^2+1}) = u_2(t)f(t-2) + 4u_2(t)g(t-2) = u_2(t)e^{-2(t-2)}\cos(t-2) + 4u_2(t)e^{-2(t-2)}\sin(t-2)$.

(h) $y''(t) - 4y'(t) + 4y(t) = \delta(t-2)$, with $y(0) = 0$ and $y'(0) = 0$.
Taking the Laplace transform $L(y''(t) + 4y'(t) + 4y(t)) = L(\delta(t-2))$, we have $(s^2 + 4s + 4)Y(s) = e^{-2s}$.
$\Rightarrow Y(s) = \frac{e^{-2s}}{(s^2+4s+4)} = \frac{e^{-2s}}{(s+2)^2}$.
Let $f(t) = L^{-1}(\frac{1}{(s+2)^2}) = te^{-2t}$.
We have $y(t) = L^{-1}(\frac{e^{-2s}}{(s+2)^2}) = u_2(t)f(t-2) = u_2(t)(t-2)e^{-2(t-2)}$.

(4) Express the solution of the given initial value problem in terms of the convolution integral.

(a) $y''(t) + 4y'(t) + 5y(t) = e^{2t}\cos(t)$, with $y(0) = 0$ and $y'(0) = 0$.
Taking the Laplace transform of the equation, we have
$L(y''(t) + 4y'(t) + 5y(t)) = L(e^{2t}\cos(t))$
$\Rightarrow (s^2 + 4s + 5)Y(s) = \frac{(s-2)}{(s-2)^2+1}$
$\Rightarrow Y(s) = \frac{(s-2)}{((s-2)^2+1)}\frac{1}{(s^2+4s+5)}$. Let $f(t) = L^{-1}(\frac{(s-2)}{(s-2)^2+1}) = e^{2t}\cos(t)$ and $g(t) = L^{-1}(\frac{1}{s^2+4s+5}) = L^{-1}(\frac{1}{(s+2)^2+1}) = e^{-2t}\sin(t)$. So $y(t) = \int_0^t f(t-\tau)g(\tau)d\tau$.

(b) $y''(t) - 2y'(t) + y(t) = te^t$, with $y(0) = 0$ and $y'(0) = 0$.
Taking the Laplace transform of the equation, we have
$L(y''(t) - 2y'(t) + y(t)) = L(te^t)$
$\Rightarrow (s^2 - 2s + 1)Y(s) = \frac{1}{(s-1)^2}$
$\Rightarrow Y(s) = \frac{1}{(s-1)^2}\frac{1}{(s^2-2s+1)} = \frac{1}{(s-1)^2} \cdot \frac{1}{(s-1)^2}$. Let $f(t) = L^{-1}(\frac{1}{(s-1)^2}) = te^t$
So $y(t) = \int_0^t f(t-\tau)f(\tau)d\tau$.

(c) $y''(t) - 3y'(t) + 2y(t) = te^t + te^{2t}$, with $y(0) = 0$ and $y'(0) = 0$.
Taking the Laplace transform of the equation, we have
$L(y''(t) - 3y'(t) + 2y(t)) = L(te^t + te^{2t})$
$\Rightarrow (s^2 - 3s + 2)Y(s) = \frac{1}{(s-1)^2} + \frac{1}{(s-2)^2}$

$\Rightarrow Y(s) = (\frac{1}{(s-1)^2} + \frac{1}{(s-2)^2})\frac{1}{(s^2-2s+1)} = (\frac{1}{(s-1)^2} + \frac{1}{(s-2)^2}) \cdot \frac{1}{(s-1)^2}$. Let $f(t) = L^{-1}(\frac{1}{(s-1)^2} + \frac{1}{(s-2)^2}) = te^t + te^{2t}$ and $g(t) = L^{-1}(\frac{1}{(s-1)^2}) = te^t$.
So $y(t) = \int_0^t f(t-\tau)g(\tau)d\tau$.