(1) Find the general solution of the following differential equations.

(a) $y^{(6)}(t) + 64y(t) = 0$.

The characteristic equation of $y^{(6)}(t) + 64y(t) = 0$ is $r^6 + 64 = 0$. Note that $-64 = 64e^{i(\pi+2k\pi)}$ where $k$ is an integer. Solving $r^6 + 64 = 0$ is the same as solving $r^6 = -64 = 64e^{i(\pi+2k\pi)}$. Therefore $r = \sqrt[6]{64}e^{i\frac{(\pi+2k\pi)}{6}} = 2e^{i\frac{(\pi+2k\pi)}{6}} = 2\cos\left(\frac{\pi+2k\pi}{6}\right) + i\sin\left(\frac{\pi+2k\pi}{6}\right)$ where $k = 0, 1, 2, \ldots, 5$.

Let $r_k = 2\cos\left(\frac{\pi+2k\pi}{6}\right) + i\sin\left(\frac{\pi+2k\pi}{6}\right)$ where $k = 0, 1, 2, \ldots, 5$. Therefore $r_0 = 2(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)) = \sqrt{3}+i$, $r_1 = 2(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)) = 2i$, $r_2 = 2(\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)) = -\sqrt{3}+i$, $r_3 = 2(\cos\left(\frac{7\pi}{6}\right)+i\sin\left(\frac{7\pi}{6}\right)) = -(\sqrt{3}+i)$, $r_4 = 2(\cos\left(\frac{11\pi}{6}\right)+i\sin\left(\frac{11\pi}{6}\right)) = \sqrt{3}-i$. So the roots of characteristic equations are $\sqrt{3} \pm i, \pm 2i$ and $-\sqrt{3} \pm i$.

The general solution is $y(t) = c_1e^{\sqrt{3}t}\cos(t) + c_2e^{\sqrt{3}t}\sin(t) + c_3\cos(2t) + c_4\sin(2t) + c_5e^{-\sqrt{3}t}\cos(t) + c_6e^{-\sqrt{3}t}\sin(t)$.

(b) $y^{(3)}(t) + 3y^{(2)}(t) + 2y'(t) = 0$.

The characteristic equation of $y^{(3)}(t) + 3y^{(2)}(t) + 2y'(t) = 0$ is $r^3 + 3r^2 + 2r - r(r^2 + 3r + 2) = r(r + 1)(r + 2) = 0$. Its roots are $r = 0, r = -1$ and $r = -2$.

The general solution is $y(t) = c_1 + c_2e^{-t} + c_3e^{-2t}$.

(c) $y^{(4)}(t) - 8y^{(2)}(t) + 16y = 0$.

The characteristic equation of $y^{(4)}(t) - 8y^{(2)}(t) + 16y = 0$ is $r^4 - 8r^2 + 16 = (r^2 - 4)^2 = (r - 2)^2(r + 2)^2$. Its roots are $r = 2$ with multiplicity 2 and $r = -2$ with multiplicity 2.

The general solution is $y(t) = c_1e^{2t} + c_2te^{2t} + c_3e^{-2t} + c_4te^{-2t}$.

Note: Compare with the problems $y^{(4)}(t) + 8y^{(2)}(t) + 16y = 0$. The characteristic equation of $y^{(4)}(t) + 8y^{(2)}(t) + 16y = 0$ is $r^4 + 8r^2 + 16 = (r^2 + 4)^2 = 0$. Its roots are $r = \pm 2i$ with multiplicity 2. The general solution is $y(t) = c_1 \cos(2t) + c_2 \sin(2t) + c_3 t \cos(2t) + c_4 t \sin(2t)$.

(d) $y^{(6)}(t) + 2y^{(3)}(t) + y(t) = 0$.

The characteristic equation of $y^{(6)}(t) + 2y^{(3)}(t) + y(t) = 0$ is $r^6 + 2r^3 + 1 = 0$.

Note that $r^6 + 2r^3 + 1 = (r^3 + 1)^2$ and $-1 = 1e^{i(\pi+2k\pi)}$ where $k$ is an integer.

Solving $r^3 + 1 = 0$ is the same as solving $r^3 = -1 = e^{i(\pi+2k\pi)}$.

Therefore $r = e^{i\frac{\pi+2k\pi}{3}} = e^{i\frac{(\pi+2k\pi)}{3}} = \cos\left(\frac{\pi+2k\pi}{3}\right) + i\sin\left(\frac{\pi+2k\pi}{3}\right)$ where $k = 0, 1, 2$.

Let $r_k = \cos\left(\frac{\pi+2k\pi}{3}\right) + i\sin\left(\frac{\pi+2k\pi}{3}\right)$ where $k = 0, 1, 2$.

Therefore $r_0 = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $r_1 = (\cos(\pi) + i\sin(\pi)) = -1$, $r_2 = 2(\cos\left(\frac{5\pi}{3}\right)+i\sin\left(\frac{5\pi}{3}\right)) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$. Note that $r_0 = \overline{r_2}$. Thus the root of $(r^3 + 1)^2$ is $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ with multiplicity 2 and $r_1 = -1$ with multiplicity 2.

The general solution is $y(t) = c_1e^{\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + c_2e^{\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right) + c_3te^{\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + c_4te^{\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right) + c_5e^{-t} + c_6te^{-t}$.
(e) \((D^2 - 4D + 13)^2(D - 2)^2y(t) = 0.\)

The characteristic equation of \((D^2 - 4D + 13)^2(D - 2)^2y(t) = 0\) is \((r^2 - 4r + 13)^2(r - 2)^2 = 0.\) Its roots are \(r = 2 \pm 3i\) with multiplicity 2 and \(r = 2\) with multiplicity 2. The general solution is \(y(t) = c_1e^{2t}\cos(3t) + c_2e^{2t}\sin(3t) + c_3te^{2t}\cos(3t) + c_4e^{2t}\sin(3t) + c_5e^{2t} + c_6te^{2t}.\)

(2) Use the method of Annihilators to find the form of particular solution of the following problems.

(a) \((D^3 - 2D^2 + D)y = t + \cos(t) + t\sin(t) + t^2e^t.\)

Solving the characteristic equation \(r^3 - 2r^2 + 2r = r(r^2 - 2r + r) = r(r - 1)^2 = 0,\) we have \(r = 0, 1, 1.\) The solutions of \((D^3 - 2D^2 + D)y = 0\) are spanned by \(1, e^t\) and \(te^t.\)

Now we should find the annihilator of \(t + \cos(t) + t\sin(t) + t^2e^t.\) We have \(D^2(t) = 0, (D^2 + 1)^2(\cos(t) + t\sin(t)) = 0\) and \((D - 1)^3(t^2e^t) = 0.\)

So \(D^2(D^2 + 1)^2(D - 1)^3(t + \cos(t) + t\sin(t) + t^2e^t) = 0.\)

The given equation is \((D^3 - 2D^2 + D)y = t + \cos(t) + t\sin(t) + t^2e^t.\) Applying the annihilator \(D^2(D^2 + 1)^2(D - 1)^3\) to the equation above, we have

\[
D^2(D^2 + 1)^2(D - 1)^3(D^3 - 2D^2 + D)y = 0.
\]

Solving the characteristic equation
\[
r^3(r^2 + 1)^2(r - 1)^3(r^3 - 2r^2 + r)
= r^3(r^2 + 1)^2(r - 1)^3 = r^3(r^2 + 1)^2(r - 1)^3 = 0,
\]
we have \(r = 0\) with multiplicity 3, \(\pm i\) with multiplicity 2, and 1 with multiplicity 5. The solution of \((D^3 - 2D^2 + D)y = t + \cos(t) + t\sin(t) + t^2e^t\) is spanned by \(1, t, t^2, \cos(t), \sin(t), t\cos(t), t\sin(t), e^t, te^t, t^2e^t, t^3e^t\) and \(t^4e^t.\) Excluding those functions \((1, e^t, te^t)\) appeared as the solution of \((D^3 + 2D^2 + D)y = 0,\) we know that the particular solution is of the form
\[
y_p(t) = c_1t + c_2t^2 + c_3\cos(t) + c_4\sin(t) + c_5t\cos(t) + c_6t\sin(t) + c_7t^2e^t + c_8te^t + c_9t^3e^t.
\]

(b) \((D^3 + D)y = t + \cos(t) + t\sin(t) + t^2e^t.\)

Solving the characteristic equation \(r^3 + r = r(r^2 + 1) = 0,\) we have \(r = 0, \pm i.\)

The solutions of \((D^3 + D)y = 0\) are spanned by 1, \(\cos(t)\) and \(\sin(t).\)

From previous question, we know that \(D^2(D^2 + 1)^2(D - 1)^3(t + \cos(t) + t\sin(t) + t^2e^t) = 0.\) So \(D^2(D^2 + 1)^2(D - 1)^3(D^3 + D)y = 0.\) Solving the characteristic equation
\[
r^3(r^2 + 1)^2(r - 1)^3(r^3 + r)
= r^3(r^2 + 1)^2(r - 1)^3 = 0,
\]
we have \(r = 0\) with multiplicity 3, \(\pm i\) with multiplicity 3, and 1 with multiplicity 2. The solution of
(D^3 + D)y = t + \cos(t) + t\sin(t) + t^2 e^t is spanned by 1, t, t^2, \cos(t), \sin(t), t \cos(t),
\sin(t), t^2 \cos(t), t^2 \sin(t), e^t, and te^t. Excluding those functions (1, \cos(t) and
\sin(t)) appeared as the solution of \((D^3 + D)y = 0\), we know that the particular
solution is of the form \(y_p(t) = c_0 t + c_1 t^2 + c_2 t \cos(t) + c_3 t \sin(t) + c_4 t^2 \cos(t) +
c_5 t^2 \sin(t) + c_6 e^t + c_7 t^2 e^t\).

(c) \(y''(t) + 2y'(t) + 2y(t) = 3te^{-t} \cos(t)\).

Solving the characteristic equation \(r^2 + 2r + 2 = 0\), we have \(r = -1 \pm i\). The
solutions of \((D^2 + 2D + 2)y = 0\) are spanned by \(e^{-t} \cos(t)\) and \(e^{-t} \sin(t)\).
The annihilator of \(te^{-t} \cos(t)\) is \((D^2 + 2D + 2)^2\). We have \((D^2 + 2D + 2)^2(te^{-t} \cos(t)) =
0\). From \((D^2 + 2D + 2)(y) = 3te^{-t} \cos(t)\), we get \((D^2 + 2D + 2)^2(D^2 + 2D + 2)(y) =
(D^2 + 2D + 2)^2(3te^{-t} \cos(t)) = 0\) and \((D^2 + 2D + 2)^3(y) = 0\). Solving the char-
acteristic equation \((r^2 + 2r + 2)^3 = 0\), we have \(r = -1 \pm i\) with multiplicity
3. The solutions of \((D^2 + 2D + 2)^3y = 0\) are spanned by \(e^{-t} \cos(t), e^{-t} \sin(t),
te^{-t} \cos(t), te^{-t} \sin(t), t^2 e^{-t} \cos(t)\) and \(t^2 e^{-t} \sin(t)\). Excluding those functions (\(e^{-t} \cos(t)\)
and \(e^{-t} \sin(t)\)) appeared as the solution of \((D^2 + 2D + 2)y = 0\), we know that
the particular solution is of the form \(y_p(t) = c_1 te^{-t} \cos(t) + c_2 te^{-t} \sin(t) +
c_3 t^2 e^{-t} \cos(t) + c_4 t^2 e^{-t} \sin(t)\).

(3) Use Laplace’s transform to find the solution of the following initial value problems.

(a) \(y^{(3)}(t) - 3y^{(2)}(t) + 2y'(t) = e^{4t}\) with \(y(0) = 1, y'(0) = 0\) and \(y''(0) = 0\).

Taking the Laplace’s transform of the equation, we have

\[
L(y^{(3)}(t) - 3y^{(2)}(t) + 2y'(t)) = L(e^{4t})
\]

\[
\Rightarrow s^3L(y) - s^2 - 3s^2L(y) + 3s + 2sL(y) - 2 = \frac{1}{s-4}
\]

\[
\Rightarrow (s^3 - 3s^2 + 2s)L(y) = s^2 - 3s + 2 + \frac{1}{s-4}
\]

\[
\Rightarrow L(y) = \frac{s^2 - 3s + 2}{(s-1)(s-2)(s-4)} + \frac{1}{(s-1)(s-2)(s-4)}
\]

\[
\Rightarrow L(y) = \frac{1}{s} + \frac{1}{(s-1)(s-2)(s-4)}
\]

Using partial fraction, we have

\[
\frac{1}{(s-1)(s-2)(s-4)} = \frac{a}{s} + \frac{b}{s-1} + \frac{c}{s-2} + \frac{d}{s-4}.
\]

Multiplying \(s(s-1)(s-2)(s-4)\), we get

\[
1 = a(s-1)(s-2)(s-4) + bs(s-2)(s-4) + cs(s-1)(s-4) + ds(s-1)(s-2).
\]

Plugging \(s = 0\), we get \(a = \frac{1}{4}\). Plugging \(s = 1\), we get \(b = \frac{1}{3}\). Plugging \(s = 2\),
we get \(c = -\frac{1}{4}\). Plugging \(s = 4\), we get \(d = \frac{1}{24}\). So we have

\[
\frac{1}{(s-1)(s-2)(s-4)} = -\frac{1}{8s} + \frac{1}{3(s-1)} - \frac{1}{4(s-2)} + \frac{1}{24(s-4)}.
\]

So we have \(L(y) = \frac{1}{s} + \frac{1}{8s} + \frac{1}{3(s-1)} - \frac{1}{4(s-2)} + \frac{1}{24(s-4)}\).
\[
= \frac{7}{8}s + \frac{1}{3}(s+1) - \frac{1}{4}(s+2) + \frac{1}{24}(s+4),
\]
and \(y(t) = L^{-1}\left(\frac{7}{8}s + \frac{1}{3}(s+1) - \frac{1}{4}(s+2) + \frac{1}{24}(s+4)\right) = \frac{7}{8} + \frac{1}{3}e^t - \frac{1}{4}e^{2t} + \frac{1}{24}e^{4t}\]
(b) \(y''(t) + y(t) = \sin(2t)\) with \(y(0) = 0, y'(0) = 0\).

Taking the Laplace’s transform and using the conditions, we have
\(L(y''(t) + y(t)) = L(\sin(2t))\)
\(\Rightarrow (s^2 + 1)Y(s) = \frac{2}{s^2+4}\)
\(\Rightarrow Y(s) = \frac{2}{(s^2+4)(s^2+1)}\)

Using partial fraction, we have \(\frac{2}{(s^2+4)(s^2+1)} = \frac{as+b}{s^2+4} + \frac{cs+d}{s^2+1}\). Multiplying \((s^2 + 4)(s^2 + 1)\), we get
\[2 = (as + b)(s^2 + 4) + (cs + d)(s^2 + 1)\]
and
\[2 = as^3 + bs^2 + 4as + 4b + cs^3 + ds^2 + cs + d = (a + c)s^3 + (b + d)s^2 + (4a + c)s + 4b + d.\]
Comparing the coefficient, we get \(a + c = 0, b + d = 0, 4a + c = 0\) and \(4b + d = 2\).
From \(a + c = 0\) and \(4a + c = 0\), we get \(a = 0\) and \(c = 0\). From \(b + d = 0\) and \(4b + d = 2\), we get \(b = \frac{2}{3}\) and \(d = -\frac{2}{3}\). So \(Y(s) = \frac{2}{(s^2+4)(s^2+1)} = \frac{2}{3}s^2 - \frac{2}{3}s^2\).
Hence \(y(t) = L^{-1}(\frac{2}{3}s^2 - \frac{2}{3}s^2) = \frac{2}{3}\sin(t) - \frac{1}{3}\sin(2t)\). Note that \(L^{-1}(\frac{1}{s^2+a^2}) = \frac{1}{a}\sin(at)\).
(c) \(y''(t) + 4y = g(t)\) with \(y(0) = 0\) and \(y'(0) = 0\) where
\[
g(t) = \begin{cases} 
0, & 0 \leq t < 2, \\
3(t - 2), & 2 \leq t < 4, \\
6, & 4 \leq t.
\end{cases}
\]
We have \(g(t) = 3(t - 2)u_{2,4}(t) + 6u_4(t) = 3(t - 2)(u_2(t) - u_4(t)) + 6u_4(t) = 3(t - 2)u_2(t) - 3(t - 2)u_2(t) = 3(t - 2)u_2(t) - 3(t - 2)u_2(t) - 3(t - 4)u_4(t).\) Let \(h(t - 2) = t - 2\) and \(k(t - 4) = t - 4\). Then \(h(t) = t\) and \(k(t) = t\). So \(g(t) = 3h(t - 2)u_2(t) - 3k(t - 4)u_4(t)\) and \(L(g(t)) = L(3h(t - 2)u_2(t) - 3k(t - 4)u_4(t)) = 3e^{-2s}L(h(t)) - 3e^{-4s}L(k(t)) = 3e^{-2s} - 3e^{-4s}\).
\(L(y''(t) + 4y(t)) = L(g(t)) = 3e^{-2s} - 3e^{-4s}\)
\(\Rightarrow (s^2 + 4)Y(s) = 3e^{-2s} - 3e^{-4s}\)
\(\Rightarrow Y(s) = \frac{3e^{-2s}}{s^2+4} - \frac{3e^{-4s}}{s^2+4}\)
\(\Rightarrow Y(s) = 3\frac{e^{-2s}}{s^2+4} - \frac{3e^{-4s}}{s^2+4}\)
Using partial fraction, we have \(\frac{1}{s^2(s^2+4)} = \frac{a}{s^2} + \frac{b}{s^2+4} + \frac{cs+d}{s^2+4}\). This implies that
\[1 = as^3(s^2 + 4) + b(s^2 + 4) + cs^3 + ds^2 = (a + c)s^3 + (b + d)s^2 + 4as + 4b.\]
Comparing the coefficient, we get \(a + c = 0, b + d = 0, 4a = 0\) and \(4b = 1\). We get \(a = 0\) and \(c = 0, b = \frac{1}{4}\) and \(d = -\frac{1}{4}\). So \(\frac{1}{s^2(s^2+4)} = \frac{1}{4s^2} - \frac{1}{4(s^2+4)}\).
\[Y(s) = 3\frac{e^{-2s}}{s^2} - 3\frac{e^{-4s}}{s^2+4} = \frac{3}{4}e^{-2s}(\frac{1}{s^2} - \frac{1}{(s^2+4)}) - \frac{3}{4}e^{-4s}(\frac{1}{s^2} - \frac{1}{(s^2+4)})\]
Let \(f(t) = L^{-1}(\frac{1}{s^2} - \frac{1}{(s^2+4)}) = t - \frac{1}{2}\sin(2t)\). Hence \(y(t) = L^{-1}\left(\frac{3}{4}e^{-2s}(\frac{1}{s^2} - \frac{1}{(s^2+4)}) - \frac{3}{4}e^{-4s}(\frac{1}{s^2} - \frac{1}{(s^2+4)})\right) = \frac{3}{4}u_2(t)f(t - 2) - \frac{3}{4}u_4(t)f(t - 4)\).
(d) \(y''(t) + 4y'(t) + 4y = g(t)\) with \(y(0) = 0\) and \(y'(0) = 0\) where

\[
g(t) = \begin{cases} 
0, & 0 \leq t < 2, \\
3(t-2), & 2 \leq t < 4, \\
6, & 4 \leq t.
\end{cases}
\]

From previous question, we have \(L(g(t)) = 3e^{-2s} - 3e^{-4s}\).

\[L(y''(t) + 4y'(t) + 4y(t)) = L(g(t)) = 3e^{-2s} - 3e^{-4s}\]

\[\Rightarrow (s^2 + 4s + 4)Y(s) = 3e^{-2s} - 3e^{-4s}\]

\[\Rightarrow Y(s) = 3e^{-2s} - \frac{3e^{-4s}}{s^2(s+2)^2} - \frac{3}{2}\]

Using partial fraction, we have \(\frac{3}{s^2(s+2)^2} = \frac{a}{s} + \frac{b}{s^2} + \frac{c(s+2) + d}{(s+2)^2}\). This implies that

\[1 = as(s+2)^2 + b(s+2)^2 + cs^2(s+2) + ds^2.\]

Plugging in \(s = 0\), we get \(b = \frac{1}{4}\). Plugging in \(s = -2\), we get \(d = \frac{1}{4}\). Hence \(1 = as(s+2)^2 + \frac{1}{4}(s+2)^2 + cs^2(s+2) + \frac{1}{4}s^2\). Plugging in \(s = 1\) and \(s = -1\), we have \(1 = 9a + \frac{9}{4} + 3c + \frac{1}{4}\) and \(1 = -a + \frac{1}{4} + c + \frac{1}{4}\). This gives \(9a + 3c = -\frac{3}{2}\) and \(-a + c = \frac{7}{2}\). Hence \(a = -\frac{1}{4}\) and \(c = \frac{1}{4}\). Now we have \(\frac{1}{s^2(s+2)^2} = -\frac{11}{4}s + \frac{1}{4}t^2 + \frac{1}{4}\) \((s+2)^2 + \frac{1}{4}\) \((s+2)^2\). Let \(f(t) = L^{-1}(\frac{11}{4}s - \frac{1}{4}t^2 + \frac{1}{4}\) \((s+2)^2 + \frac{1}{4}\) \((s+2)^2\)) = \(-\frac{11}{4} + \frac{1}{4}t + \frac{1}{4}e^{-2t}\). Then \(y(t) = L^{-1}(\frac{3}{s^2(s+2)^2} - \frac{e^{-4s}}{s^2(s+2)^2}) = 3u_2(t)f(t-2) - 3u_4(t)f(t-4)\).

(e) \(y''(t) + 4y'(t) + 5y = g(t)\) with \(y(0) = 0\) and \(y'(0) = 0\) where

\[
g(t) = \begin{cases} 
0, & 0 \leq t < 2, \\
1, & 2 \leq t < 4, \\
0, & 4 \leq t.
\end{cases}
\]

We have \(g(t) = u_{2,4}(t) = u_2(t) - u_4(t)\) and \(L(g(t)) = e^{-2s} - e^{-4s}\). Taking the Laplace transform, we get \(L(y''(t) + 5y'(t) + 5y) = L(g(t))\) and \((s^2 + 4s + 5)Y(s) = e^{-2s} - e^{-4s}\).

\[\Rightarrow Y(s) = \frac{e^{-2s}}{(s^2 + 4s + 5)} - \frac{e^{-4s}}{(s^2 + 4s + 5)}.\]

Note that \(\frac{1}{(s^2 + 4s + 5)} = \frac{1}{(s+2)^2 + 1}\) and \(f(t) = L^{-1}(\frac{1}{(s+2)^2 + 1}) = e^{-2t}\sin(t)\).

Then \(y(t) = L^{-1}(\frac{e^{-2s}}{(s^2 + 4s + 5)} - \frac{e^{-4s}}{(s^2 + 4s + 5)}) = u_2(t)f(t-2) - u_4(t)f(t-4)\).

(f) \(y''(t) + 5y'(t) + 4y(t) = \delta(t-2)\), with \(y(0) = 0\) and \(y'(0) = 0\).

Taking the Laplace transform \(L(y''(t) + 5y'(t) + 4y(t)) = L(\delta(t-2))\), we have \((s^2 + 5s + 4)Y(s) = e^{-2s}\).

\[\Rightarrow Y(s) = \frac{e^{-2s}}{(s+1)(s+4)} = e^{-2s}(\frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s+4}) = \frac{1}{3}\frac{e^{-2s}}{s+1} - \frac{1}{3}\frac{e^{-2s}}{s+4}.
\]

Let \(f(t) = L^{-1}(\frac{1}{s+1}) = e^{-t}\) and \(g(t) = L^{-1}(\frac{1}{s+4}) = e^{-4t}\).
We have \( y(t) = L^{-1}\left(\frac{1}{3}e^{-2s} - \frac{1}{3}s e^{-2s}\right) = \frac{1}{3}u_2(t)f(t-2) - \frac{1}{3}u_2(t)g(t-2) = \frac{1}{3}u_2(t)e^{-(t-2)} - \frac{1}{3}u_2(t)e^{-4(t-2)}\).

(g) \( y''(t) + 4y'(t) + 5y(t) = \delta(t-2) \), with \( y(0) = 1 \) and \( y'(0) = 1 \).

Taking the Laplace transform \( L(y''(t) + 4y'(t) + 5y(t)) = L(\delta(t-2)) \), we have
\[
(s^2 + 4s + 5)Y(s) - s - 5 = e^{-2s}.
\]

\[\Rightarrow Y(s) = \frac{e^{-2s} (s + 5)}{(s^2 + 4s + 5)} + \frac{e^{-2s}}{(s^2 + 4s + 5)} + \frac{e^{-2s}}{(s + 2)} + \frac{4(e^{-2s})}{(s + 2)^2 + 1}\]

Let \( f(t) = L^{-1}\left(\frac{e^{-2s}}{(s + 2)^2 + 1}\right) = e^{-2t} \cos(t) \) and \( g(t) = L^{-1}\left(\frac{1}{(s + 2)^2 + 1}\right) = e^{-2t} \sin(t) \)

We have \( y(t) = L^{-1}\left(\frac{1}{(s + 2)^2 + 1}\right) = u_2(t)f(t-2) + 4u_2(t)g(t-2) = u_2(t)e^{-2(t-2)} \cos(t-2) + 4u_2(t)e^{-2(t-2)} \sin(t-2)\).

(h) \( y''(t) - 4y'(t) + 4y(t) = \delta(t-2) \), with \( y(0) = 0 \) and \( y'(0) = 0 \).

Taking the Laplace transform \( L(y''(t) + 4y'(t) + 4y(t)) = L(\delta(t-2)) \), we have
\[
(s^2 + 4s + 4)Y(s) = e^{-2s}.
\]

\[\Rightarrow Y(s) = \frac{e^{-2s}}{(s + 2)^2} = \frac{e^{-2s}}{(s + 2)^2}.
\]

Let \( f(t) = L^{-1}\left(\frac{1}{(s + 2)^2}\right) = te^{-2t} \).

We have \( y(t) = L^{-1}\left(\frac{e^{-2s}}{(s + 2)^2}\right) = u_2(t)f(t-2) = u_2(t)(t-2)e^{-2(t-2)}\).

(4) Express the solution of the given initial value problem in terms of the convolution integral.

(a) \( y''(t) + 4y'(t) + 5y(t) = e^{2t} \cos(t), \) with \( y(0) = 0 \) and \( y'(0) = 0 \).

Taking the Laplace transform of the equation, we have
\[L(y''(t) + 4y'(t) + 5y(t)) = L(e^{2t} \cos(t))\]
\[\Rightarrow (s^2 + 4s + 5)Y(s) = \frac{1}{(s + 2)^2 + 1}\]
\[\Rightarrow Y(s) = \frac{1}{(s + 2)^2 + 1} \cdot \frac{1}{(s^2 + 4s + 5)} \cdot \frac{1}{(s - 2)^2 + 1}
\]

Let \( f(t) = L^{-1}\left(\frac{1}{(s + 2)^2 + 1}\right) = e^{2t} \cos(t) \) and \( g(t) = L^{-1}\left(\frac{1}{(s - 2)^2 + 1}\right) = e^{-2t} \sin(t) \). So \( y(t) = \int_0^t f(t - \tau)g(\tau)d\tau \).

(b) \( y''(t) - 2y'(t) + y(t) = te^t, \) with \( y(0) = 0 \) and \( y'(0) = 0 \).

Taking the Laplace transform of the equation, we have
\[L(y''(t) - 2y'(t) + y(t)) = L(te^t)\]
\[\Rightarrow (s^2 - 2s + 1)Y(s) = \frac{1}{(s - 1)^2}\]
\[\Rightarrow Y(s) = \frac{1}{(s - 1)^2} \cdot \frac{1}{(s^2 - 2s + 1)} = \frac{1}{(s - 1)^2} \cdot \frac{1}{(s - 1)^2} \cdot \frac{1}{(s - 1)^2}
\]

Let \( f(t) = L^{-1}\left(\frac{1}{(s - 1)^2}\right) = te^t \)

So \( y(t) = \int_0^t f(t - \tau)g(\tau)d\tau \).

(c) \( y''(t) - 3y'(t) + 2y(t) = te^t + te^{2t}, \) with \( y(0) = 0 \) and \( y'(0) = 0 \).

Taking the Laplace transform of the equation, we have
\[L(y''(t) - 3y'(t) + 2y(t)) = L(te^t + te^{2t})\]
\[\Rightarrow (s^2 - 3s + 2)Y(s) = \frac{1}{(s - 1)^2} + \frac{1}{(s - 2)^2}\]
\[ Y(s) = \left( \frac{1}{(s-1)^2} + \frac{1}{(s-2)^2} \right) \left( \frac{1}{s^2-2s+1} \right) = \left( \frac{1}{(s-1)^2} + \frac{1}{(s-2)^2} \right) \cdot \frac{1}{(s-1)^2}. \]

Let \( f(t) = L^{-1}\left( \frac{1}{(s-1)^2} + \frac{1}{(s-2)^2} \right) = te^t + te^{2t} \) and \( g(t) = L^{-1}\left( \frac{1}{(s-1)^2} \right) = te^t. \)

So \( y(t) = \int_0^t f(t - \tau)g(\tau)d\tau. \)