(1) Use Laplace transform to find the solution of the following problems

(a) $y'' + 4y' + 3y = 0$ with $y(0) = 2$ and $y'(0) = -1$.

(b) $y'' + 4y' + 5y = 0$ with $y(0) = 2$ and $y'(0) = -1$.

(c) $y^4(t) - 16y = 0$ with $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$ and $y'''(0) = 1$.

Solution:

(a) Taking the Laplace's transform of the differential equation $y'' + 4y' + 3y = 0$,
we have $L(y'' + 4y' + 3y) = L(0) = 0$.
Using $L(y'') = s^2L(y) - sy(0) - y'(0)$ and $L(y') = sL(y) - y(0)$, we have
$s^2L(y) - sy(0) - y'(0) + 4(sL(y) - y(0)) + 3L(y) = 0$ and
$(s^2 + 4s + 3)L(y) - sy(0) - y'(0) - 4y(0) = 0$.
Substituting $y(0) = 2$ and $y'(0) = -1$, we have
$(s^2 + 4s + 3)L(y) - s \cdot 2 - (-1) - 4 \cdot 2 = 0$ and
$(s^2 + 4s + 3)L(y) = 2s + 7$. So we have $L(y) = \frac{2s+7}{s^2+4s+3}$.

Note that $s^2 + 4s + 3 = (s + 1)(s + 3)$. We can use partial fraction to get $\frac{2s+3}{s^2+4s+3} = \frac{2s+7}{(s+1)(s+3)} = \frac{a}{s+1} + \frac{b}{s+3}$. To find a and b, we multiply the equation by $(s + 1)(s + 3)$ to get $2s+7 = a(s+3)+b(s+1) = as+3a+bs+b = (a+b)s+3a+b$. Comparing the coefficient, we have $a+b = 2$ and $3a+b = 7$. Solving $a + b = 2$ and $3a + b = 7$, we get $a = \frac{5}{2}$ and $b = -\frac{1}{2}$. This implies that $\frac{2s+7}{s^2+4s+3} = \frac{5}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s+3}$.

From $L(y) = \frac{2s+7}{s^2+4s+3} = \frac{5}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s+3}$, we get
$y(t) = L^{-1}(\frac{5}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s+3}) = \frac{5}{2}L^{-1}(\frac{1}{s+1}) - \frac{1}{2}L^{-1}(\frac{1}{s+3})$
$= \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$.

(b) Taking the Laplace's transform of the differential equation $y'' + 4y' + 5y = 0$,
we have $L(y'' + 4y' + 5y) = L(0) = 0$.
Using $L(y'') = s^2L(y) - sy(0) - y'(0)$ and $L(y') = sL(y) - y(0)$, we have
$s^2L(y) - sy(0) - y'(0) + 4(sL(y) - y(0)) + 5L(y) = 0$ and
$(s^2 + 4s + 5)L(y) - sy(0) - y'(0) - 4y(0) = 0$.
Substituting $y(0) = 2$ and $y'(0) = -1$, we have
$(s^2+4s+5)L(y) - s \cdot 2 - (-1) - 4 \cdot 2 = 0$ and $(s^2+4s+5)L(y) = 2s + 3$. So we have $L(y) = \frac{2s+7}{s^2+4s+5}$.

Note that $s^2 + 4s + 5 = (s + 2)^2 + 1$. We can use the substitution $u = s + 2$ and $s = u - 2$ to get $\frac{2s+7}{s^2+4s+3} = \frac{2s+7}{(s+2)^2+1} = \frac{2(u-2)+7}{u^2+1} = \frac{2u+3}{u^2+1} = 2\frac{u}{u^2+1} + \frac{3}{u^2+1} = 2\frac{(s+2)}{(s+2)^2+1} + \frac{3}{(s+2)^2+1}$.

From $L(y)=\frac{2s+3}{s^2+4s+5}=2\frac{(s+2)}{(s+2)^2+1}+\frac{3}{(s+2)^2+1}$, we get $y(t)=L^{-1}(2\frac{(s+2)}{(s+2)^2+1}+\frac{3}{(s+2)^2+1})=2L^{-1}(\frac{(s+2)}{(s+2)^2+1})+3L^{-1}(\frac{1}{(s+2)^2+1})=2e^{-2t}\cos(t)+3e^{-2t}\sin(t)$. Note that we have used the fact that $L^{-1}(\frac{(s+2)}{(s+2)^2+1})=e^{-2t}\cos(t)$ and $L^{-1}(\frac{1}{(s+2)^2+1})=e^{-2t}\sin(t)$.

(c) Taking the Laplace's transform of the differential equation $y^{(4)}-16y=0$, we have $L(y^{(4)}-16y)=0$. Using $L(y'''')=s^4L(y)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)$, we have $s^4L(y)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)-16L(y)=0$ and $(s^4-16)L(y)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)=0$. Substituting $y(0)=1$, $y'(0)=0$, $y''(0)=1$, $y'''(0)=1$ we have $(s^4-16)L(y)-s^3-s-1=0$, $(s^4-16)L(y)=s^3+s+1$. So we have $L(y)=\frac{s^3+s+1}{s^4-16}$. Note that $s^4-16=(s^2-4)(s^2+4)$. We can rewrite $\frac{s^3+s+1}{s^4-16}$ as $\frac{s^3+s+1}{s^4-16}=\frac{s^3+s+1}{(s^2-4)(s^2+4)}=\frac{as+b}{(s^2-4)}+\frac{cs+d}{(s^2+4)}$. Multiply $(s^2-4)(s^2+4)$, we get
$s^3+s+1=(as+b)(s^2+4)+(cs+d)(s^2-4)$
$=as^3+bs^2+4as+4b+cs^3+ds^2-4cs-4d$
$=(a+c)s^3+(b+d)s^2+(4a-4c)s+4b-4d$. Comparing the coefficient, we have $a+c=1$, $b+d=0$, $4a-4c=1$ and $4b-4d=1$. Solving $a+c=1$ and $4a-4c=1$, we get $a=\frac{5}{8}$ and $c=\frac{3}{8}$ Solving $b+d=0$ and $4b-4d=1$, we get $b=\frac{1}{8}$ and $d=-\frac{1}{8}$. Hence $\frac{s^3+s+1}{s^4-16}=\frac{s^3+s+1}{(s^2-4)(s^2+4)}=\frac{\frac{5}{8}s+\frac{1}{8}}{(s^2-4)}+\frac{\frac{3}{8}s-\frac{1}{8}}{(s^2+4)}=\frac{5}{8}\frac{s}{(s^2-4)}+\frac{1}{8}\frac{1}{(s^2-4)}+\frac{3}{8}\frac{s}{(s^2+4)}-\frac{1}{8}\frac{1}{(s^2+4)}$. Using $L^{-1}(\frac{s}{(s^2-4)})=\cosh(2t)$, $L^{-1}(\frac{1}{(s^2-4)})=\frac{1}{2}\sinh(2t)$, $L^{-1}(\frac{s}{(s^2+4)})=\cos(2t)$, $L^{-1}(\frac{1}{(s^2+4)})=\frac{1}{2}\sin(2t)$, we have $y(t)=\frac{5}{8}L^{-1}(\frac{s}{(s^2-4)})+\frac{1}{8}L^{-1}(\frac{1}{(s^2-4)})+\frac{3}{8}L^{-1}(\frac{s}{(s^2+4)})-\frac{1}{8}L^{-1}(\frac{1}{(s^2+4)})=\frac{1}{8}\cosh(2t)+\frac{3}{8}\cdot\frac{1}{2}\sinh(2t)-\frac{3}{8}\cos(t)-\frac{1}{8}\cdot\frac{1}{2}\sin(t)=\frac{1}{8}\cosh(2t)+\frac{3}{16}\sinh(2t)-\frac{3}{8}\cos(t)-\frac{1}{16}\sin(t)$.