(1) Use Laplace transform to find the solution of the following problems

(a) $y''+4y'+5y = \cos(2t)+\sin(2t)$ with $y(0) = 2$ and $y'(0) = -1$.

(b) $y'' + 4y' + 4y = 3e^{-2t}$ with $y(0) = 2$ and $y'(0) = -1$.

Solution:

(a) Taking the Laplace's transform of the differential equation
$y'' + 4y' + 5y = \cos(2t) + \sin(2t)$,we have
$L(y'' + 4y' + 5y) = L(\cos(2t) + \sin(2t)) = \frac{s}{s^2+4} + \frac{2}{s^2+4} = \frac{s+2}{s^2+4}$.
Using $L(y'') = s^2L(y)-sy(0)-y'(0)$ and $L(y') = sL(y)-y(0)$, we have
$s^2L(y) - sy(0) - y'(0) + 4(sL(y) - y(0)) + 5L(y) = \frac{s+2}{s^2+4}$ and
$(s^2 + 4s + 5)L(y) - sy(0) - y'(0) - 4y(0) = \frac{s+2}{s^2+4}$.
Substituting $y(0) = 2$ and $y'(0) = -1$, we have
$(s^2 + 4s + 5)L(y) - s \cdot 2 - (-1) - 4 \cdot 2 = \frac{s+2}{s^2+4}$ and
$(s^2 + 4s + 5)L(y) = 2s + 7 + \frac{s+2}{s^2+4}$.
So $L(y) = \frac{2s+7}{s^2+4s+5} + \frac{s+2}{(s^2+4s+5)(s^2+4)}$.

Note that $s^2 + 4s + 5 = (s + 2)^2 + 1$. First, we simplify the term $\frac{2s+7}{s^2+4s+5}$. We can use the substitution $u = s + 2$ and $s = u - 2$ to get
$\frac{2s+7}{s^2+4s+5} = \frac{2s+7}{(s+2)^2+1} = \frac{2(u-2)+7}{u^2+1} = \frac{2u+3}{u^2+1}$
$= 2\frac{u}{u^2+1} + \frac{3}{u^2+1} = 2\frac{(s+2)}{(s+2)^2+1} + \frac{3}{(s+2)^2+1}$.

Now we simplify the term $\frac{s+2}{(s^2+4s+5)(s^2+4)}$ by partial fraction. We have
$\frac{s+2}{(s^2+4s+5)(s^2+4)} = \frac{s+2}{((s+2)^2+1)(s^2+4)} = \frac{a(s+2)+b}{(s+2)^2+1} + \frac{cs+d}{s^2+4}$. Multiplying $((s + 2)^2 + 1)(s^2 + 4)$, we have
$s + 2 = (a(s + 2) + b)(s^2 + 4) + (cs + d)((s + 2)^2 + 1)$
$= (as + (2a + b))(s^2 + 4) + (cs + d)(s^2 + 4s + 5)$
$= as^3+(2a+b)s^2+4as+(8a+4b)+cs^3+ds^2+4cs^2+4ds+5cs+5d$
$= (a+c)s^3+((2a+b)+d+4c)s^2+(4a+4d+5c)s+(8a+4b+5d)$.
Comparing the coefficient, we get $a+c = 0$, $2a+b+4c+d = 0$, $4a + 5c + 4d = 1$ and $8a + 4b + 5d = 2$. From $a + c = 0$, we have $c = -a$, $-2a+b+d = 0$, $-a+4d = 1$ and $8a+4b+5d = 2$. Multiplying -4 to $-2a + b + d = 0$, we get $8a - 4b - 4d = 0$. Adding $8a + 4b + 5d = 2$, we have $16a + d = 2$. Using $-a + 4d = 1$ and $16a + d = 2$, we have $a = \frac{7}{65}$ and $d = \frac{18}{65}$. From $-2a + b + d = 0$ and $c = -a$, we get $b = 2a - d = \frac{-4}{65}$

and $c = -\frac{7}{65}$. Hence
$\frac{s+2}{(s^2+4s+5)(s^2+4)} = \frac{7}{65}\frac{(s+2)}{(s+2)^2+1} - \frac{4}{65}\frac{1}{(s+2)^2+1} - \frac{7}{65}\frac{s}{s^2+4} + \frac{18}{65}\frac{1}{s^2+4}$.

From $L(y) = \frac{2s+7}{s^2+4s+5} + \frac{s+2}{(s^2+4s+5)(s^2+4)}$,
$\frac{2s+7}{s^2+4s+5} = 2\frac{(s+2)}{(s+2)^2+1} + \frac{3}{(s+2)^2+1}$ and
$\frac{s+2}{(s^2+4s+5)(s^2+4)} = \frac{7}{65}\frac{(s+2)}{(s+2)^2+1} - \frac{4}{65}\frac{1}{(s+2)^2+1} - \frac{7}{65}\frac{s}{s^2+4} + \frac{18}{65}\frac{1}{s^2+4}$, we get
$L(y) = \frac{137}{65}\frac{(s+2)}{(s+2)^2+1} + \frac{191}{65}\frac{1}{(s+2)^2+1} - \frac{7}{65}\frac{s}{s^2+4} + \frac{18}{65}\frac{1}{s^2+4}$ and $y(t) = \frac{137}{65}L^{-1}(\frac{(s+2)}{(s+2)^2+1}) + \frac{191}{65}L^{-1}(\frac{1}{(s+2)^2+1}) - \frac{7}{65}L^{-1}(\frac{s}{s^2+4}) + \frac{18}{65}L^{-1}(\frac{1}{s^2+4}) = \frac{137}{65}e^{-2t}\cos(t) + \frac{191}{65}e^{-2t}\sin(t) - \frac{7}{65}\cos(2t) + \frac{9}{65}\sin(2t)$. Note that we have used the fact that $L^{-1}(\frac{(s+2)}{(s+2)^2+1}) = e^{-2t}\cos(t)$, $L^{-1}(\frac{1}{(s+2)^2+1}) = e^{-2t}\sin(t)$, $L^{-1}(\frac{s}{(s^2+4)}) = \cos(2t)$ and $L^{-1}(\frac{1}{s^2+4}) = \frac{1}{2}\sin(2t)$.

(b)

Taking the Laplace's transform of the differential equation $y'' + 4y' + 4y = 3e^{-2t}$,
we have $L(y''+4y'+4y) = L(3e^{-2t})$. Using $L(y'') = s^2L(y) - sy(0) - y'(0)$, $L(y') = sL(y) - y(0)$ and $L(3e^{-2t}) = 3L(e^{-2t}) = \frac{3}{s+2}$, we have
$s^2L(y) - sy(0) - y'(0) + 4(sL(y) - y(0)) + 4L(y) = \frac{3}{s+2}$ and
$(s^2 + 4s + 4)L(y) - sy(0) - y'(0) - 4y(0) = \frac{4}{s+2}$.
Substituting $y(0) = 2$ and $y'(0) = -1$, we have
$(s^2+4s+4)L(y) - s\cdot 2 - (-1) - 4\cdot 2 = \frac{3}{s+2}$ and $(s^2+4s+4)L(y) = \frac{3}{s+2} + 2s + 7$. Note that $s^2 + 4s + 4 = (s+2)^2$. So we have $(s+2)^2L(y) = \frac{3}{s+2} + 2s + 7$ and $L(y) = \frac{3}{(s+2)^3} + \frac{2s+7}{(s+2)^2}$. We can simplify $\frac{2s+7}{(s+2)^2}$ by substituting $u = s+2$, $s = u-2$ and $\frac{2s+7}{(s+2)^2} = \frac{2(u-2)+7}{u^2} = \frac{2u+3}{u^2} = \frac{2}{u} + \frac{3}{u^2} = \frac{2}{(s+2)} + 3\frac{1}{(s+2)^2}$.

Hence we have $L(y) = \frac{3}{(s+2)^3} + \frac{2}{(s+2)} + 3\cdot\frac{1}{(s+2)^2}$ and
$y = L^{-1}(\frac{3}{(s+2)^3} + \frac{2}{(s+2)} + 3\frac{1}{(s+2)^2})$.
Using $L(e^{-2t}) = \frac{1}{s+2}$, $L(te^{-2t}) = \frac{1}{(s+2)^2}$, and $L(t^2e^{-t}) = \frac{2}{(s+2)^3}$, we have $L^{-1}(\frac{1}{s+2}) = e^{-2t}$, $L^{-1}(\frac{1}{(s+2)^2}) = te^{-2t}$ and $L^{-1}(\frac{1}{(s+2)^3}) = \frac{t^2e^{-t}}{2}$.
Therefore
$y = L^{-1}(\frac{3}{(s+2)^3} + \frac{2}{(s+2)} + \frac{3}{(s+2)^2})$
$= 3L^{-1}(\frac{1}{(s+2)^3}) + 2L^{-1}(\frac{1}{(s+2)}) + 3L^{-1}(\frac{1}{(s+2)^2})$
$= 3\cdot\frac{t^2e^{-2t}}{2} + 2e^{-2t} + 3te^{-t} = \frac{3}{2}t^2e^{-2t} + 2e^{-2t} + 3te^{-2t}$.