(1) Use Laplace's transform to find the solution of the following initial value problems.

$y''(t)+4y'(t)+5y = h(t)$ with $y(0) = 1$ and $y'(0) = 1$ where

$$h(t) = \begin{cases} 0, & 0 \le t < 3, \\ t-3, & 3 \le t < 6, \\ 3, & 6 \le t. \end{cases}$$

Solution:

First, we find the Laplace's transform of $h(t)$.

$$h(t) = \begin{cases} 0, & 0 \le t < 3, \\ t-3, & 3 \le t < 6, \\ 0, & 6 \le t. \end{cases} + \begin{cases} 0, & 0 \le t < 3, \\ 0, & 3 \le t < 6, \\ 3, & 6 \le t. \end{cases}$$

$$= (t-3) \cdot \begin{cases} 0, & 0 \le t < 3, \\ 1, & 3 \le t < 6, \\ 0, & 3 \le t. \end{cases} + 3 \cdot \begin{cases} 0, & 0 \le t < 3, \\ 0, & 2 \le t < 6, \\ 1, & 3 \le 6. \end{cases}$$

$$= (t-3)u_{3,6}(t)+3u_6(t) = (t-3)(u_3(t)-u_6(t))+3u_6(t) = (t-3)u_3(t)+(-t+6)u_6(t)$$

$= (t-3)u_3(t) - (t-6)u_6(t)$.

Let $k(t-3) = t-3$ and $l(t-6) = t-6$. Then $h(t) = k(t-3)u_3(t) - l(t-6)u_6(t)$, $k(t) = t$ and$l(t) = t$.

Hence $L(h(t)) = L(k(t-3)u_3(t) - l(t-6)u_6(t))$

$= L(k(t-3)u_3(t)) - L(l(t-6)u_6(t))$

$= e^{-3s}L(k(t)) - e^{-6s}L(l(t)) = e^{-3s}L(t) - e^{-6s}L(t) = \frac{e^{-3s}}{s^2} - \frac{e^{-6s}}{s^2}$.

Taking the Laplace's transform of the differential equation $y'' + 4y' + 5y = h(t)$,we have

$L(y'' + 4y' + 5y) = L(h(t)) = \frac{e^{-3s}}{s^2} - \frac{e^{-6s}}{s^2}$. Using $L(y'') = s^2L(y) - sy(0) - y'(0)$ and $L(y') = sL(y) - y(0)$, we have

$s^2L(y) - sy(0) - y'(0) + 4(sL(y) - y(0)) + 5L(y) = \frac{e^{-3s}}{s^2} - \frac{e^{-6s}}{s^2}$

and $(s^2 + 4s + 5)L(y) - sy(0) - y'(0) - 4y(0) = \frac{e^{-3s}}{s^2} - \frac{e^{-6s}}{s^2}$.

Substituting $y(0) = 1$ and $y'(0) = 1$, we have

$(s^2 + 4s + 5)L(y) - s \cdot 1 - 1 - 4 = \frac{e^{-3s}}{s^2} - \frac{e^{-6s}}{s^2}$ and

$(s^2 + 4s + 5)L(y) = s + 5 + \frac{e^{-3s}}{s^2} - \frac{e^{-6s}}{s^2}$.

So $L(y) = \frac{s+5}{s^2+4s+5} + \frac{e^{-3s}}{(s^2+4s+5)s^2} - \frac{e^{-6s}}{(s^2+4s+5)s^2}$.

Note that $s^2 + 4s + 5 = (s+2)^2 + 1$. First, we simplify the term $\frac{s+5}{s^2+4s+5}$. We have

$$(1) \quad \frac{s+5}{s^2+4s+5} = \frac{(s+2)+3}{(s+2)^2+1} = \frac{(s+2)}{(s+2)^2+1} + \frac{3}{(s+2)^2+1}.$$

Now we simplify the term $\frac{1}{(s^2+4s+5)s^2}$ by partial fraction. We have
$\frac{1}{(s^2+4s+5)s^2} = \frac{1}{((s+2)^2+1)s^2} = \frac{a(s+2)+b}{(s+2)^2+1} + \frac{cs+d}{s^2}$. Multiplying $((s+2)^2+1)s^2$, we have
$1 = (a(s+2)+b)s^2 + (cs+d)((s+2)^2+1)$
$= (as+(2a+b))s^2 + (cs+d)(s^2+4s+5)$
$= as^3 + (2a+b)s^2 + cs^3 + ds^2 + 4cs^2 + 4ds + 5cs + 5d$
$= (a+c)s^3 + ((2a+b)+d+4c)s^2 + (4d+5c)s + 5d$. Comparing the coefficient, we get $a+c=0$, $2a+b+4c+d=0$, $5c+4d=0$ and $5d=1$. From $5d=1$, we have $d=\frac{1}{5}$. From $5c+4d=0$, we have $5c=-4d=-\frac{4}{5}$ and $c=-\frac{4}{25}$. From $a+c=0$, we have $a=-c=\frac{4}{25}$. Form $2a+b+4c+d=0$, we have $b=-2a-4c-d=-2\cdot\frac{4}{25}-4\cdot\frac{-4}{25}-\frac{1}{5}=\frac{3}{25}$.
This implies that

$$\frac{1}{(s^2+4s+5)s^2} = \frac{\frac{4}{25}(s+2)+\frac{3}{25}}{(s+2)^2+1} + \frac{-\frac{4}{25}s+\frac{1}{5}}{s^2} \tag{2}$$

$$= \frac{4}{25}\frac{(s+2)}{(s+2)^2+1} + \frac{3}{25}\frac{1}{(s+2)^2+1} - \frac{4}{25}\frac{1}{s} + \frac{1}{5}\frac{1}{s^2}. \tag{3}$$

From equations (1) and (3), we have
$L(y) = \frac{s+5}{s^2+4s+5} + \frac{e^{-3s}}{(s^2+4s+5)s^2} - \frac{e^{-6s}}{(s^2+4s+5)s^2}$
$= \frac{(s+2)}{(s+2)^2+1} + \frac{3}{(s+2)^2+1} + e^{-3s}\cdot(\frac{4}{25}\frac{(s+2)}{(s+2)^2+1} + \frac{3}{25}\frac{1}{(s+2)^2+1} - \frac{4}{25}\frac{1}{s} + \frac{1}{5}\frac{1}{s^2})$
$-e^{-6s}\cdot(\frac{4}{25}\frac{(s+2)}{(s+2)^2+1} + \frac{3}{25}\frac{1}{(s+2)^2+1} - \frac{4}{25}\frac{1}{s} + \frac{1}{5}\frac{1}{s^2})$.

Let $f(t) = L^{-1}(\frac{4}{25}\frac{(s+2)}{(s+2)^2+1} + \frac{3}{25}\frac{1}{(s+2)^2+1} - \frac{4}{25}\frac{1}{s} + \frac{1}{5}\frac{1}{s^2})$
$= \frac{4}{25}e^{-2t}\cos(t) + \frac{3}{25}e^{-2t}\sin(t) - \frac{4}{25} + \frac{1}{5}t$.
Then $L^{-1}(e^{-3s}\cdot(\frac{4}{25}\frac{(s+2)}{(s+2)^2+1} + \frac{3}{25}\frac{1}{(s+2)^2+1} - \frac{4}{25}\frac{1}{s} + \frac{1}{5}\frac{1}{s^2})$
$= u_3(t)f(t-3)$ and
$L^{-1}(e^{-6s}\cdot(\frac{4}{25}\frac{(s+2)}{(s+2)^2+1} + \frac{3}{25}\frac{1}{(s+2)^2+1} - \frac{4}{25}\frac{1}{s} + \frac{1}{5}\frac{1}{s^2})$
$= u_6(t)f(t-6)$. Note that we have used $L^{-1}(e^{-cs}F(s)) = u_c(t)f(t-c)$ where $f(t) = L^{-1}(F(s))$.

Finally, we get
$y(t)$
$= L^{-1}\left(\frac{(s+2)}{(s+2)^2+1} + \frac{3}{(s+2)^2+1} + (e^{-3s} - e^{-6s})\cdot(\frac{4}{25}\frac{(s+2)}{(s+2)^2+1} + \frac{3}{25}\frac{1}{(s+2)^2+1} - \frac{4}{25}\frac{1}{s} + \frac{1}{5}\frac{1}{s^2})\right)$
$= e^{-2t}\cos(t) + 3e^{-2t}\sin(t) + u_3(t)f(t-3) - u_6(t)f(t-6)$
where $f(t) = \frac{4}{25}e^{-2t}\cos(t) + \frac{3}{25}e^{-2t}\sin(t) - \frac{4}{25} + \frac{1}{5}t$.