Solution to HW 8

1. ((Sec 3.6 Problem 2))
Solution:Solving $r^2 + 2r + 5 = 0$,we have $r = -1 \pm 2i$. The solution of $y''(t) + 2y'(t) + 5y = 0$ is $y(t) = c_1e^{-t}\sin(2t) + c_2e^{-t}\cos(2t)$. We try $y_p = c\sin(2t) + d\cos(2t)$ to be a particular solution of $y''(t) + 2y'(t) + 5y = 3\sin(2t)$. We have $y_p = c\sin(2t) + d\cos(2t)$,
$y_p' = 2c\cos(2t) - 2d\sin(2t)$,
$y_p'' = -4c\sin(2t) - 4d\cos(2t)$ and $y_p''(t) + 2y_p'(t) + 5y_p(t) = -4c\sin(2t) - 4d\cos(2t) + 4c\cos(2t) - 4d\sin(2t) + 5c\sin(2t) + 5d\cos(2t) = (c-4d)\sin(2t) + (4c+d)\cos(2t) = 3\sin(2t)$ if $c-4d = 3$, $4c+d = 0$, $c = \frac{3}{17}$ and $d = -\frac{12}{17}$. Thus the general solution of $y''(t) + 2y'(t) + 5y = 3\sin(2t)$ is $y(t) = \frac{3}{17}\sin(2t) + -\frac{12}{17}\cos(2t) + c_1e^{-t}\sin(2t) + c_2e^{-t}\cos(2t)$.

2. ((Sec 3.6 Problem 14)) Solving $r^2+4 = 0$, we know that the solution of $y''(t)+4y = 0$ is $y(t) = c_1\sin(2t)+c_2\cos(2t)$. We try $y_p = ct^2+dt+e+fe^t$ to be a particular solution of $y''(t)+4y = t^2+3e^t$. We have $y_p = ct^2+dt+e+fe^t$,
$y_p' = 2ct + d + fe^t$,
$y_p'' = 2c + fe^t$ and $y_p''(t) + 4y_p(t) = 2c + fe^t + 4ct^2 + 4dt + 4e + 4fe^t = 4ct^2 + 4dt + (2c + 4e) + 5fe^t = t^2 + 3e^t$ if $4c = 1$, $4d = 0$, $2c + 4e = 0$ and $5f = 3$. So $c = \frac{1}{4}$, $d = 0$, $e = -\frac{1}{8}$ and $f = \frac{3}{5}$.

Thus the general solution of is $y(t) = \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t + c_1\sin(2t) + c_2\cos(2t)$. Using the initial condition $y(0) = 0$ and $y'(0) = 2$, we get $y(0) = -\frac{1}{8} + \frac{3}{5} + c_2 = 0$, $y'(0) = \frac{3}{5} + 2c_1 = 2$, $c_1 = \frac{7}{10}$ and $c_2 = -\frac{19}{40}$. Thus $y(t) = \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t + \frac{7}{10}\sin(2t) - \frac{19}{40}\cos(2t)$.

3. ((Sec 3.6 Problem 17)) Solution: Solving $r^2 + 4 = 0$, we know that the solution of $y''(t) + 4y = 0$ is $y(t) = c_1\sin(2t) + c_2\cos(2t)$. We try $y_p = ct\sin(2t)+dt\cos(2t)$ to be a particular solution of $y''(t)+4y = 3\sin(2t)$. We have $y_p = ct\sin(2t) + dt\cos(2t)$,
$y_p' = c\sin(2t) + 2ct\cos(2t) + d\cos(2t) - 2dt\sin(2t)$,
$y_p'' = 4c\cos(2t) - 4ct\sin(2t) - 4d\sin(2t) - 4dt\cos(2t)$ and $y_p''(t) + 4y_p(t) = 4c\cos(2t) - 4d\sin(2t) = 3\sin(2t)$ if $c = 0$ and $d = -\frac{3}{4}$. Thus the general solution of $y''(t)+4y = 3\sin(2t)$ is $y(t) = -\frac{3}{4}t\cos(2t)+c_1\sin(2t)+c_2\cos(2t)$. Using the condition $y(0) = 2$ and $y'(0) = -1$, we get $c_1 = -\frac{1}{8}$ and $c_2 = 2$. Thus $y(t) = -\frac{3}{4}t\cos(2t) - \frac{1}{8}\sin(2t) + 2\cos(2t)$.

4. (Sec 3.7 Problem 5)
Solution: Solving $r^2 + 1 = 0$, we have $r = \pm i$. So the general solution of $y''(t) + y(t) = 0$ is $y(t) = c_1\cos(t) + c_2\sin(t)$. We will use the variation of parameter formula to solve $y''(t) + y(t) = \tan(t)$. We have $y_1(t) = \cos(t)$, $y_2(t) = \sin(t)$, $g(t) = \tan(t)$
$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = \cos(t)\cdot(\cos(t)) - \sin(t)\cdot(-\sin(t)) =$

$\cos^2(t) + \sin^2(t) = 1$,
$\int \frac{y_2 g(t)}{W(y_1,y_2)(t)} dt = \int \frac{\sin(t)\tan(t)}{1} dt = \int \frac{\sin^2(t)}{\cos(t)} dt = \int \frac{1-\cos^2(t)}{\cos(t)} dt = \int \sec(t) - \cos(t) dt = \ln|\sec(t) + \tan(t)| - \sin(t) + c$ and
$\int \frac{y_1 g(t)}{W(y_1,y_2)(t)} dt = \int \frac{\cos(t)\tan(t)}{1} dt = \int \sin(t) dt = -\cos(t) + d$. We have Thus $y(t) = -\cos(t) \cdot (\ln|\sec(t) + \tan(t)| - \sin(t) + c) + \sin(t)(-\cos(t) + d)$.

5. (Sec 3.7 Problem 14)
Solution: Rewrite $t^2 y''(t) - t(t+2)y'(t) + (t+2)y(t) = 2t^3$ as
$y''(t) - \frac{t+2}{t} y'(t) + \frac{t+2}{t^2} y(t) = 2t$.
We will use the variation of parameter formula to solve$y''(t) - \frac{t+2}{t} y'(t) + \frac{t+2}{t^2} y(t) = 2t$. We have $y_1(t) = t$, $y_2(t) = te^t$, $g(t) = 2t$
$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = t(e^t + te^t) - te^t \cdot 1 = t^2 e^t$,
$\int \frac{y_2 g(t)}{W(y_1,y_2)(t)} dt = \int \frac{te^t \cdot 2t}{t^2 e^t} dt = \int 2dt = 2t + c$ and
$\int \frac{y_1 g(t)}{W(y_1,y_2)(t)} dt = \int \frac{t \cdot 2t}{t^2 e^t} dt = \int 2e^{-t} dt = -2e^{-t} + d$. We have Thus $y(t) = -t \cdot (2t + c) + te^t(-2e^{-t} + d) = -2t^2 - ct - 2t + dte^t = -2t^2 - (c+2)t + dte^t$.
So $y_p(t) = -2t^2$ or $y_p(t) = -2t^2 - 2t$.