Solution to HW 8

1. (Sec 3.6 Problem 2]) Solution: Solving \( r^2 + 2r + 5 = 0 \), we have \( r = -1 \pm 2i \). The solution of \( y''(t) + 2y'(t) + 5y = 0 \) is \( y(t) = e^{-t} \sin(2t) + e^{-t} \cos(2t) \). We try \( y_p = c \sin(2t) + d \cos(2t) \) to be a particular solution of \( y''(t) + 2y'(t) + 5y = 3 \sin(2t) \). We have \( y_p = c \sin(2t) + d \cos(2t) \),
\[
y_p' = 2c \cos(2t) - 2d \sin(2t),
\]
\[
y_p'' = -4c \sin(2t) - 4d \cos(2t) \quad \text{and} \quad y_p''(t) + 2y_p'(t) + 5y_p(t) = -4c \sin(2t) - 4d \cos(2t) + 4c \cos(2t) - 4d \sin(2t) + 5c \sin(2t) + 5d \cos(2t) = (c - 4d) \sin(2t) + (4c + d) \cos(2t) = 3 \sin(2t) \]
if \( c - 4d = 3 \), \( 4c + d = 0 \), \( c = \frac{3}{17} \) and \( d = -\frac{12}{17} \). Thus the general solution of \( y''(t) + 2y'(t) + 5y = 3 \sin(2t) \) is \( y(t) = \frac{3}{17} \sin(2t) + -\frac{12}{17} \cos(2t) + c e^{-t} \sin(2t) + c_2 e^{-t} \cos(2t) \).

2. (Sec 3.6 Problem 14]) Solving \( r^2 + 4 = 0 \), we know that the solution of \( y''(t) + 4y = 0 \) is \( y(t) = c_1 \sin(2t) + c_2 \cos(2t) \). We try \( y_p = c t^2 + dt + e + f e^t \) to be a particular solution of \( y''(t) + 4y = t^2 + 3e^t \). We have \( y_p = c t^2 + dt + e + f e^t \),
\[
y_p' = 2c t + d + f e^t \quad \text{and} \quad y_p''(t) + 4y_p(t) = 2c + f e^t + 4ct^2 + 4dt + 4e + 4fe^t = 4ct^2 + 4dt + (2c + 4e) + 5fe^t = t^2 + 3e^t \]
if \( 4c = 1 \), \( 4d = 0 \), \( 2c + 4e = 0 \) and \( 5f = 3 \). So \( c = \frac{1}{4} \), \( d = 0 \), \( e = -\frac{3}{8} \) and \( f = \frac{3}{8} \).

Thus the general solution of is \( y(t) = \frac{1}{4} t^2 - \frac{1}{8} + \frac{3}{8} e^t + c_1 \sin(2t) + c_2 \cos(2t) \). Using the initial condition \( y(0) = 0 \) and \( y'(0) = 2 \), we get \( y(0) = -\frac{1}{8} + \frac{3}{8} + c_2 = 0 \), \( y'(0) = \frac{3}{8} + 2c_1 = 2 \), \( c_1 = \frac{7}{10} \) and \( c_2 = -\frac{19}{40} \). Thus \( y(t) = \frac{1}{4} t^2 - \frac{1}{8} + \frac{3}{8} e^t + \frac{7}{10} \sin(2t) - \frac{19}{40} \cos(2t) \).

3. (Sec 3.6 Problem 17]) Solution: Solving \( r^2 + 4 = 0 \), we know that the solution of \( y''(t) + 4y = 0 \) is \( y(t) = c_1 \sin(2t) + c_2 \cos(2t) \). We try \( y_p = c t \sin(2t) + dt \cos(2t) \) to be a particular solution of \( y''(t) + 4y = 3 \sin(2t) \). We have \( y_p = c t \sin(2t) + dt \cos(2t) \),
\[
y_p' = c \sin(2t) + 2ct \cos(2t) + d \cos(2t) - 2dt \sin(2t),
\]
\[
y_p'' = 4c \cos(2t) - 4ct \sin(2t) - 4d \sin(2t) - 4dt \cos(2t) \quad \text{and} \quad y_p''(t) + 4y_p(t) = 4c \cos(2t) - 4d \sin(2t) = 3 \sin(2t) \]
if \( c = 0 \) and \( d = -\frac{3}{4} \). Thus the general solution of \( y''(t) + 4y = 3 \sin(2t) \) is \( y(t) = -\frac{3}{4} t \cos(2t) + c_1 \sin(2t) + c_2 \cos(2t) \).

Using the condition \( y(0) = 2 \) and \( y'(0) = -1 \), we get \( c_1 = -\frac{1}{8} \) and \( c_2 = 2 \). Thus \( y(t) = -\frac{3}{4} t \cos(2t) - \frac{1}{8} \sin(2t) + 2 \cos(2t) \).

4. (Sec 3.7 Problem 5)
Solution: Solving \( r^2 + 1 = 0 \), we have \( r = \pm i \). So the general solution of \( y''(t) + y(t) = 0 \) is \( y(t) = c_1 \cos(t) + c_2 \sin(t) \). We will use the variation of parameter formula to solve \( y''(t) + y(t) = \tan(t) \). We have \( y_1(t) = \cos(t) \), \( y_2(t) = \sin(t) \), \( g(t) = \tan(t) \)
\[
W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = \cos(t) \cdot (\cos(t)) - \sin(t) \cdot (-\sin(t)) = \]
\[
\frac{1}{\cos(t)}.
\]
\[
\cos^2(t) + \sin^2(t) = 1,
\]
\[
\int \frac{y_2(t)}{W(y_1, y_2)(t)} dt = \int \frac{\sin(t) \tan(t)}{1} dt = \int \frac{\sin^2(t)}{\cos(t)} dt = \int \frac{1 - \cos^2(t)}{\cos(t)} dt = \int \sec(t) - \cos(t) dt = \ln |\sec(t) + \tan(t)| - \sin(t) + c \quad \text{and}
\]
\[
\int \frac{y_1(t)}{W(y_1, y_2)(t)} dt = \int \frac{\cos(t) \tan(t)}{1} dt = \int \sin(t) dt = -\cos(t) + d. \quad \text{We have Thus}
\]
\[y(t) = -\cos(t) \cdot (\ln |\sec(t) + \tan(t)| - \sin(t) + c) + \sin(t)(-\cos(t) + d).
\]

5. (Sec 3.7 Problem 14)

Solution: Rewrite \(t^2 y''(t) - t(t+2)y'(t) + (t+2)y(t) = 2t^3\) as
\[y''(t) - \frac{t+2}{t} y'(t) + \frac{t^2}{t^2} y(t) = 2t. \quad \text{We have } y_1(t) = t, \ y_2(t) = te^t, \ g(t) = 2t
\]
\[W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = t(e^t + te^t) - te^t \cdot 1 = t^2 e^t,
\]
\[
\int \frac{y_2(t)}{W(y_1, y_2)(t)} dt = \int \frac{te^t}{t^2 e^t} dt = \int 2 dt = 2t + c \quad \text{and}
\]
\[
\int \frac{y_1(t)}{W(y_1, y_2)(t)} dt = \int \frac{2t}{t^2 e^t} dt = \int 2e^{-t} dt = -2e^{-t} + d. \quad \text{We have Thus } y(t) = -t \cdot (2t + c) + te^t(-2e^{-t} + d) = -2t^2 - ct - 2t + dt e^t = -2t^2 - (c + 2)t + dt e^t.
\]
So \(y_p(t) = -2t^2\) or \(y_p(t) = -2t^2 - 2t\).