Solution to Quiz #4 and HW 5

1. (quiz problem, (Sec 2.6 Problem 3)) Determine if the following equation is exact. Find the solution if it is exact.

$$(3x^2 - 2xy + 2)dx + (6y^2 - x^2 + 3)dy = 0$$

Solution: Let $M(x, y) = 3x^2 - 2xy + 2$ and $N(x, y) = 6y^2 - x^2 + 3$. We have $M_y = -2x$ and $N_x = -2x$. So $M_y = N_x$ and the given equation is exact. Thus there is a function $\phi(x, y)$ such that $\phi_x = M = 3x^2 - 2xy + 2$ and $\phi_y = N = 6y^2 - x^2 + 3$. Integrating the first equation, we have $\phi(x, y) = \int(3x^2 - 2xy + 2)dx = x^3 - x^2y + 2x + h(y)$. Using $\phi_y = N = 6y^2 - x^2 + 3$, we have $\frac{\partial(x^3 - x^2y + 2x + h(y))}{\partial y} = 6y^2 - x^2 + 3$, $-x^2 + h'(y) = 6y^2 - x^2 + 3$, $h'(y) = 6y^2 + 3$ and $h(y) = \int(6y^2 + 3)dy = 2y^3 + 3y$. Hence $\phi(x, y) = x^3 - x^2y + 2x + h(y) = x^3 - x^2y + 2x + 2y^3 + 3y$ and the solution satisfies $\phi(x, y) = x^3 - x^2y + 2x + 2y^3 + 3y = c$.

2. (Sec 2.6 Problem 4) $(2xy^2 + 2y) + (2x^2y + 2x)y' = 0$
Solution: Let $M(x, y) = 2xy^2 + 2y$ and $N(x, y) = 2x^2y + 2x$. We have $M_y = 4xy + 2$ and $N_x = 4xy + 2$. So $M_y = N_x$ and the given equation is exact. Thus there is a function $\phi(x, y)$ such that $\phi_x = M = 2xy^2 + 2y$ and $\phi_y = N = 2x^2y + 2x$. Integrating the first equation, we have $\phi(x, y) = \int(2xy^2 + 2y)dx = x^2y^2 + 2xy + h(y)$. Using $\phi_y = N = 2x^2y + 2x$, we have $\frac{\partial(x^2y^2 + 2xy + h(y))}{\partial y} = 2x^2y + 2x$, $2x^2y + 2x + h'(y) = 2x^2y + 2x$, $h'(y) = 0$ and $h(y) = c$. Hence $\phi(x, y) = x^2y^2 + 2xy + h(y) = x^2y^2 + 2xy + c$ and the solution satisfies $\phi(x, y) = x^2y^2 + 2xy = C$.

3. (Sec 2.6 Problem 16) Determine the value of b such that the following equation is exact, and then solve the equation. $(ye^{2xy} + x)dx + bxe^{2xy}dy = 0$.
Solution: Let $M(x, y) = ye^{2xy} + x$ and $N(x, y) = bxe^{2xy}$. We have $M_y = e^{2xy} + 2xye^{2xy}$ and $N_x = be^{2xy} + 2bxye^{2xy}$. So $M_y = N_x$ if $b = 1$. Thus there is a function $\phi(x, y)$ such that $\phi_x = M = ye^{2xy} + x$ and $\phi_y = N = xe^{2xy}$. Integrating the first equation, we have $\phi(x, y) = \int(ye^{2xy} + x)dx = \frac{e^{2xy}}{2} + \frac{x^2}{2} + h(y)$. Using $\phi_y = N = xe^{2xy}$ (b/c $b = 1$), we have $\frac{\partial(\frac{e^{2xy}}{2} + \frac{x^2}{2} + h(y))}{\partial y} = xe^{2xy}$, $xe^{2xy} + h'(y) = xe^{2xy}$, $h'(y) = 0$ and $h(y) = c$. Hence $\phi(x, y) = \frac{e^{2xy}}{2} + \frac{x^2}{2} + h(y) = \frac{e^{2xy}}{2} + \frac{x^2}{2} + c$ and the solution satisfies $\phi(x, y) = \frac{e^{2xy}}{2} + \frac{x^2}{2} = C$.

4. (Sec 3.1 Problem 10) $y'' + 4y' + 3y = 0$, $y(0) = 2$ and $y'(0) = -1$.
Solution: Solving $r^2 + 4r + 3 = 0$, we have $r = -1$ and $r = -3$. So $y(t) = ce^{-t} + de^{-3t}$. Using $y(0) = 2$, we have $c + d = 2$. Now $y'(t) = -ce^{-t} - 3de^{-3t}$ Using $y'(0) = -1$, we have $-c - 3d = -1$. Solving $c + d = 2$ and $-c - 3d = -1$, we have $d = -\frac{1}{2}$ and $c = \frac{5}{2}$. Hence $y(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$.

5. (Sec 3.1 Problem 21)
Solution: Solving $r^2 - r - 2 = 0$, we have $r = 2$ and $r = -1$. So $y(t) = ce^{2t} + de^{-t}$. Using $y(0) = \alpha$, we have $c + d = \alpha$. Now $y'(t) = 2ce^{2t} - de^{-t}$ Using $y'(0) = 2$, we have $2c - d = 2$. Solving $c + d = \alpha$ and $2c - d = -2$, we have $c = \frac{\alpha-2}{3}$ and $d = \frac{2+2\alpha}{3}$. Hence $y(t) = (\frac{\alpha-2}{3})e^{2t} + (\frac{2+2\alpha}{3})e^{-t}$. Now $\lim_{t\to\infty} e^{2t} = \infty$ and $\lim_{t\to\infty} e^{-t} = 0$. So $\lim_{t\to\infty} y(t) = 0$ if $\frac{\alpha-2}{3} = 0$, that is $\alpha = 2$.

6. (Sec 3.1 Problem 24)
Solution:Solution: Solving $r^2 + (3 - \alpha) - 2(\alpha - 1) = (r + 2)(r - (\alpha - 1)$, we have $r = -2$ and $r = \alpha - 1$. So $y(t) = ce^{-2t} + de^{(\alpha-1)t}$. Now $\lim_{t\to\infty} e^{-2t} = 0$ and $\lim_{t\to\infty} e^{(\alpha-1)t} = 0$ if $\alpha - 1 < 0$. So $\alpha < 1$. One can conclude that if $\alpha < 1$ then $\lim_{t\to\infty} y(t) = 0$.

Since $\lim_{t\to\infty} e^{-2t} = 0$, so $\lim_{t\to\infty} y(t) = 0$. if $d = 0$. So we can't find α such that all the solutions become unbounded.