Solution to Quiz #5 and HW 6

1. (quiz problem, (Sec 3.2 Problem 9))
Solution: Rewrite the equation as $y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$. The coefficients are not continous at $t = 0$ and $t = 4$. Since $t_0 = 3 \in (0, 4)$, the largest interval of existence is $0 < t < 4$.

2. (quiz problem, (Sec 3.3 Problem 15))
Solution: Rewrite the equation as $y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 0$. Let $p(t) = -\frac{t+2}{t} = -1 - \frac{2}{t}$. Now the Wronskian is $W(t) = ce^{-\int p(t)dt} = ce^{-\int(-1-\frac{2}{t})dt} = ce^{\int(1+\frac{2}{t})dt} = ce^{t+2\ln t} = ce^t e^{2\ln t} = ce^t t^2$.

3. (Sec 3.2 Problem 16) Solution: Since $y(t) = \sin(t^2)$, we have $y(0) = 0$, $y'(t) = 2t\cos(t^2)$ and $y'(0) = 0$, By the uniqueness of the solution of homogeneous equation, we must have $y(t) = 0$. This means that $y(t) = \sin(t^2)$ can't be a solution of $y'' + p(t)y' + q(t)y = 0$.

4. (Sec 3.2 Problem 25) Solution: Since $y_1(x) = x$ and $y_2(x) = xe^x$, we have $y_1'(x) = 1$, $y_1''(x) = 0$, $y_2'(x) = e^x + xe^x$and $y_2''(x) = e^x + e^x + xe^x = 2e^x + xe^x$. So $x^2y_1'' - x(x+2)y_1' + (x+2)y_1 = 0 - x(x+2) + (x+2)x = 0$ and $x^2y_2'' - x(x+2)y_2' + (x+2)y_2 = x^2(2e^x + xe^x) - x(x+2)(e^x + xe^x) + (x+2)xe^x = 2x^2e^x + x^3e^x - x^2e^x - x^3e^x - 2xe^x - 2x^2e^x + x^2e^x + 2xe^x = 0$.

So y_1 and y_2 are solutions. The Wronskain $W(x) = y_1y_2' - y_2y_1' = x \cdot (e^x + xe^x) - xe^x \cdot 1 = x^2e^x \neq 0$ if $x > 0$. Therefore y_1 and y_2 forms a fundamental set of solutions.

5. (Sec 3.3 Problem 18)
Solution: Rewrite the equation as $y'' - \frac{2x}{1-x^2}y' + \frac{\alpha(\alpha+1)}{1-x^2}y = 0$. Let $p(x) = -\frac{2x}{1-x^2}$. Now the Wronskian is $W(x) = ce^{-\int p(x)dx} = ce^{-\int(-\frac{2x}{1-x^2})dx} = ce^{\int(\frac{2x}{1-x^2})dx} = ce^{-\ln(1-x^2)} = \frac{c}{1-x^2}$. We have used the substitution $u = 1 - x^2$ and $du = -2xdx$ to get $\int(\frac{2x}{1-x^2})dx = -\int \frac{du}{u} = -\ln(u) + c = -\ln(1-x^2) + c$.

6. (Sec 3.3 Problem 24)
Solution: Suppose y_1 and y_2 have zero at $t = t_0$, then the Wronskian $W(t_0) = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) = 0$. By Abel Theorem, we know that $W(t) = 0$ Thus y_1 and y_2 can't be a set of fundamental solutions.

7. (Sec 3.4 Problem 11)
Solution:Solving $r^2 + 6r + 13 = 0$, we have $r = -3 \pm 2$. Thus $y(t) = c_1e^{-3t}\cos(2t) + c_2e^{-3t}\sin(2t)$.