Solution to Quiz #6 and HW 7

1. (quiz problem, (Sec 3.4 Problem 41))
Solution:uppose $y(t) = t^r$, we have $y'(t) = rt^{r-1}$ and $y''(t) = r(r-1)t^{r-2}$. Thus $t^2y''(t)+3ty'(t)+1.25y(t) = (r(r-1)+3r+1.25)t^r = (r^2+2r+1.25)t^r$. So $y = t^r$ is a solution of $t^2y''(t)+3ty'(t)+1.25y(t) = 0$ if $r^2+2r+1.25 = 0$ and $r = \frac{-2\pm\sqrt{4-5}}{2} = -1 \pm \frac{i}{2}$. Now $t^{(-1\pm\frac{i}{2})t} = t^{-1}e^{(\frac{i}{2}\ln t)} = t^{-1}\cos(\frac{\ln t}{2}) + it^{-1}\sin(\frac{\ln t}{2})$. So $y(t) = c_1t^{-1}\cos(\frac{\ln t}{2}) + c_2t^{-1}\sin(\frac{\ln t}{2})$ is the general solution of $t^2y''(t) + 3ty'(t) + 1.25y(t) = 0$

2. ((Sec 3.4 Problem 42))
Solution:uppose $y(t) = t^r$, we have $y'(t) = rt^{r-1}$ and $y''(t) = r(r-1)t^{r-2}$. Thus $t^2y''(t)-4ty'(t)-6y(t) = (r(r-1)-4r-6)t^r = (r^2-5r-6)t^r$. So $y = t^r$ is a solution of $t^2y''(t)-4ty'(t)-6y(t) = 0$ if $r^2-5r-6 = (r-6)(r+1) = 0$, $r = 6$ and $r = -1$. So $y(t) = c_1t^6 + c_2t^{-1}$ is the general solution of $t^2y''(t) - 4ty'(t) - 6y(t) = 0$

3. ((Sec 3.5 Problem 12))
The characteristic equation is $r^2 - 6r + 9 = 0$. Note that $r^2 - 6r + 9 = (r-3)^2$. It has repeated roots $r = 3$. The general solution of $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 0$ is $y(t) = c_1e^{3t} + c_2te^{3t}$. Thus $y'(t) = 3c_1e^{3t} + c_2e^{3t} + 3c_2te^{3t} = (3c_1 + c_2)e^{3t} + 3c_2te^{3t}$. Using $y(0) = 0$ and $y'(0) = 2$, we have $c_1 = 0$ and $3c_1 + 4c_2 = 2$. Solving $c_1 = 0$ and $3c_1 + 4c_2 = 2$, we get $c_1 = 0$ and $c_2 = 2$. Hence $y(t) = 2te^{3t}$. We also have $\lim_{t\to\infty} 2te^{3t} = \infty$.

4. ((Sec 3.5 Problem 28))
Solution: Rewrite the equation $(x-1)y'' - xy' + y = 0$ as $y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = 0$ Let y_2 be another solution of $(x-1)y'' - xy' + y = 0$. We have $(\frac{y_2}{y_1})' = \frac{y_1y_2' - y_1'y_2}{y_1^2} = \frac{W(x)}{y_1^2} = \frac{Ce^{\int \frac{x}{x-1}dx}}{(e^x)^2} = \frac{Ce^{\int(1+\frac{1}{x-1})dx}}{e^{2x}} = \frac{Ce^{(x+\ln(x-1))}}{e^{2x}} = \frac{Ce^x(x-1)}{e^{2x}} = C(x-1)e^{-x}$. So $\frac{y_2}{y_1} = \int C(x-1)e^{-x} = -Cxe^{-x} + D$ and $y_2 = y_1(-Cxe^{-x} + D) = e^x(-Cxe^{-x} + D) = -Cx + De^x$. So the second solution is x.

5. (Sec 3.6 Problem 1)
Solution: Solving $r^2 - 2r - 3 = (r-3)(r+1) = 0$The solution of $y''(t) - 2y'(t) - 3y(t) = 0$ is $y(t) = c_1e^{3t} + c_2e^{-t}$. We try $y_p = ce^{2t}$ to be a particular solution of $y''(t) - 2y'(t) - 3y(t) = 3e^{2t}$. We have $y_p' = 2ce^{2t}$, $y_p'' = 4ce^{2t}$ and $y_p''(t) - 2y_p'(t) - 3y_p(t) = 4ce^{2t} - 4ce^{2t} - 3ce^{2t} = -3ce^{2t}$. So $y_p''(t) - 2y_p'(t) - 3y_p(t) = 3e^{2t}$ if $-3c = 3$ and $c = -1$. Thus the general solution of $y''(t) - 2y'(t) - 3y(t) = 3e^{2t}$ is $y(t) = -e^{2t} + c_1e^{3t} + c_2e^{-t}$.

6. (Sec 3.6 Problem 12)
Solution: Solving $r^2 - r - 2 = (r-2)(r+1) = 0$, we know that the solution of $y''(t) - y'(t) - 2y(t) = 0$ is $y(t) = c_1e^{2t} + c_2e^{-t}$. We try $y_p = cte^{2t} + de^{-2t}$ to

be a particular solution of $y''(t) - y'(t) - 2y(t) = \cosh(2t) = \frac{e^{2t}+e^{-2t}}{2}$. We have $y_p = cte^{2t} + de^{-2t}$, $y_p' = ce^{2t} + 2cte^{2t} - 2de^{-2t}$, $y_p'' = 2ce^{2t} + 2ce^{2t} + 4cte^{2t} + 4de^{-2t} = 4ce^{2t} + 4cte^{2t} + 4de^{-2t}$ and $y_p''(t) - y_p'(t) - 2y_p(t) = (4ce^{2t} + 4cte^{2t} + 4de^{-2t}) - (ce^{2t} + 2cte^{2t} - 2de^{-2t}) - 2(cte^{2t} + de^{-2t}) = (4c - c)e^{2t} + (4c - 2c - 2c)te^{2t} + (4d + 2d - 2d)e^{-2t} = 3ce^{2t} + 4de^{-2t}$. So $y_p''(t) - y_p'(t) - 2y_p(t) = \frac{e^{2t}+e^{-2t}}{2}$ if $3c = \frac{1}{2}$, $4d = \frac{1}{2}$, $c = \frac{1}{6}$ and $d = \frac{1}{8}$. Thus the general solution of $y''(t) - y'(t) - 2y(t) = \cosh(2t)$ is $y(t) = \frac{1}{6}te^{2t} + \frac{1}{8}e^{-2t} + c_1e^{2t} + c_2e^{-t}$.