Solution to Quiz #7 and HW 9

1. Quiz problem (Sec 4.3 Problem 18)

The given equation can be rewritten as

$$(D^4 + 2D^3 + 2D^2)y = 3e^t + 2te^{-t} + e^{-t}\sin(t).$$

First we solve $(D^4+2D^3+2D^2)y = 0$. Solving the characteristic equation $r^4+2r^3+2r^2 = r^2(r^2+2r+2) = r^2((r+1)^2+1) = 0$, we have $r = 0,\ 0,\ -1\pm i$. The solutions of $(D^4 + 2D^3 + 2D^2)y = 0$ are spanned by 1, t, $e^{-t}\cos(t)$ and $e^{-t}\sin(t)$.

Now we should find the annihilator of $3e^t + 2te^{-t} + e^{-t}\sin(t)$. We have $(D-1)(e^t) = 0$, $(D+1)^2(te^{-t}) = 0$ and $((D+1)^2+1)(e^{-t}\sin(t)) = 0$. So $(D-1)(D+1)^2((D+1)^2+1)(3e^t + 2te^{-t} + e^{-t}\sin(t)) = 0$.

The given equation is $(D^4+2D^3+2D^2)y = 3e^t+2te^{-t}+e^{-t}\sin(t)$. Applying the annihilator $(D-1)(D+1)^2((D+1)^2+1)$ to the equation above, we have

$$\begin{aligned}&(D-1)(D+1)^2((D+1)^2+1)(D^4+2D^3+2D^2)y\\ &= (D-1)(D+1)^2((D+1)^2+1)(3e^t+2te^{-t}+e^{-t}\sin(t)) = 0.\end{aligned}$$

Solving the characteristic equation
$(r-1)(r+1)^2(((r+1)^2+1)(r^4+2r^3+2r^2)$
$= (r-1)(r+1)^2(((r+1)^2+1)r^2((r+1)^2+1) = (r-1)(r+1)^2(((r+1)^2+1)^2r^2 = 0$, we have $r = 1,\ -1,\ -1,\ -1\pm i,\ -1\pm i,\ 0$ and 0. The solution of $(D^4 + 2D^3 + 2D^2)y = 3e^t + 2te^{-t} + e^{-t}\sin(t)$ is spanned by e^t, e^{-t}, te^{-t}, $e^{-t}\cos(t)$, $e^{-t}\sin(t)$, $te^{-t}\cos(t)$, $te^{-t}\sin(t)$, 1 and t. Excluding those functions appeared as the solution of $(D^4 + 2D^3 + 2D^2)y = 0$, we know that the particular solution is of the form $y_p(t) = c_1e^t + c_2e^{-t} + c_3te^{-t} + c_4te^{-t}\cos(t) + c_5te^{-t}\sin(t)$.

2. (Sec 4.1 Problem 6) We rewrite the equation $(x^2-4)y^6(t) + x^2y''' + 9y = 0$ as $y^6(t) + \frac{x^2}{x^2-4}y''' + \frac{9}{x^2-4}y = 0$. Now the function $\frac{x^2}{x^2-4}$ and $\frac{9}{x^2-4}$ are continuous on $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$. So the solution exists on $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$.

3. (Sec 4.2 Problem 15) The characteristic equation of $y^{(6)}(t) + y = 0$ is $r^6 + 1 = 0$. So $r^6 = -1 = e^{i(\pi+2k\pi)}$ where k is an integer. Now $r = e^{\frac{i(\pi+2k\pi)}{6}}$ for $k = 0$, 1, 2, 3, 4 and 5. So $r = e^{i(\frac{\pi}{6})} = \cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} + i\frac{1}{2}$, $r = e^{i(\frac{3\pi}{6})} = e^{i(\frac{\pi}{2})} = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) = i$, $r = e^{i(\frac{5\pi}{6})} == \cos(\frac{5\pi}{6}) + i\sin(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$, $r = e^{i(\frac{7\pi}{6})} == \cos(\frac{7\pi}{6}) + i\sin(\frac{7\pi}{6}) = -\frac{\sqrt{3}}{2} - i\frac{1}{2}$, $r = e^{i(\frac{9\pi}{6})} == \cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2}) = -i$, $r = e^{i(\frac{11\pi}{6})} == \cos(\frac{11\pi}{6}) + i\sin(\frac{11\pi}{6}) = \frac{\sqrt{3}}{2} - i\frac{1}{2}$. So $r = \frac{\sqrt{3}}{2} \pm i\frac{1}{2}$, $\pm i$ and $r = -\frac{\sqrt{3}}{2} \pm i\frac{1}{2}$. Thus the general solution is $y(t) = c_1e^{\frac{\sqrt{3}t}{2}}\cos(\frac{t}{2}) + c_2e^{\frac{\sqrt{3}t}{2}}\sin(\frac{t}{2}) + c_3\cos(t) + c_4\sin(t) + c_5e^{\frac{-\sqrt{3}t}{2}}\cos(\frac{t}{2}) + c_6e^{\frac{-\sqrt{3}t}{2}}\sin(\frac{t}{2})$.

4. (Sec 4.3 Problem 1) The given equation can be rewritten as $(D^3 - D^2 - D + 1)y = 2e^{-t} + 3$. First we solve $(D^3 - D^2 - D + 1)y = 0$. Solving the characteristic equation $r^3 - r^2 - r + 1 = r^2(r-1) - (r-1) = (r^2-1)(r-1) = (r-1)^2(r+1) = 0$, we have $r = 1$, 1 and -1. The solutions of $(D^3 - D^2 - D + 1)y = 0$ are spanned by e^t, te^t and e^{-t}.

Now we should find the annihilator of $2e^{-t}+3$. We have $(D+1)(e^{-t}) = 0$, $D(3) = 0$. So $(D+1)D(2e^{-t}+3) = 0$.

The given equation is $(D^3 - D^2 - D + 1)y = 2e^{-t} + 3$. Applying the annihilator $(D+1)D$ to the equation above, we have $(D+1)D(D^3 - D^2 - D + 1)y = (D+1)D(2e^{-t}+3) = 0$. Solving the characteristic equation $(r-1)r(r^3-r^2-r+1) = (r-1)r(r-1)^2(r+1) = r(r-1)^3(r+1) = 0$, we have $r = 0, 1, 1, 1$ and -1. The solution of $D^3 - D^2 - D + 1)y = 2e^{-t} + 3$ is spanned by 1, e^t, te^t, t^2e^t and e^{-t}. Excluding those functions appeared as the solution of $(D^3 - D^2 - D + 1)y = 0$, we know that the particular solution is of the form $y_p(t) = c_1 + c_2t^2e^t$. With $y_p(t) = c_1 + c_2t^2e^t$, we have $y_p'(t) = 2c_2te^t + c_2t^2e^t$, $y_p''(t) = 2c_2e^t + 2c_2te^t + 2c_2te^t + c_2t^2e^t = 2c_2e^t + 4c_2te^t + c_2t^2e^t$ and $y_p'''(t) = 2c_2e^t + 4c_2e^t + 4c_2te^t + 2c_2te^t + c_2t^2e^t = 6c_2e^t + 6c_2te^t + c_2t^2e^t$. So $y_p''' - y_p'' - y_p' + y = 6c_2e^t + 6c_2te^t + c_2t^2e^t - (2c_2e^t + 4c_2te^t + c_2t^2e^t) - (2c_2te^t + c_2t^2e^t) + c_1 + c_2t^2e^t = 2c_2e^t + c_1 = 2e^{-t} + 3$ if $c_1 = 3$ and $c_2 = 1$. Thus the general solution of $(D^3 - D^2 - D + 1)y = 2e^{-t} + 3$ is $y(t) = 3 + t^2e^{-t} + c_1e^t + c_2te^t + c_3e^{-t}$.

5. (Sec 4.3 Problem 13)

The given equation can be rewritten as $(D^3 - 2D^2 + D)y = t^3 + 2e^t$. First we solve $(D^3 - 2D^2 + D)y = 0$. Solving the characteristic equation $r^3 - 2r^2 + r = r(r^2 - 2r + 1) = r(r-1)^2 = 0$, we have $r = 0$, 1 and 1. The solutions of $(D^3 - 2D^2 + D)y = 0$ are spanned by 1, e^t and te^t.

Now we should find the annihilator of $t^3 + 2e^t$.

We have $D^4(t^3) = 0$, $(D-1)(e^t) = 0$. So $D^4(D-1)(t^3 + 2e^t) = 0$.

The given equation is $(D^3 - 2D^2 + D)y = t^3 + 2e^t$. Applying the annihilator $D^4(D-1)$ to the equation above, we have $D^4(D-1)(D^3 - 2D^2 + D)y = D^4(D-1)(t^3 + 2e^t) = 0$. Solving the characteristic equation $r^4(r-1)(r^3 - 2r^2 + r) = r^4(r-1)r(r-1)^2 = r^5(r-1)^3 = 0$, we have $r = 0$ with multiplicities 5 and $r = 1$ with multiplicities 3. The solution of $(D^3 - 2D^2 + D)y = t^3 + 2e^t$ is spanned by 1, t, t^2, t^3, t^4, e^t, te^t and t^2e^t. Excluding those functions appeared as the solution of $(D^3 - 2D^2 + D)y = 0$, we know that the particular solution is of the form $y_p(t) = c_1t + c_2t^2 + c_3t^3 + c_4t^4 + c_5t^2e^t$.

6. (Sec 4.3 Problem 17)

The given equation can be rewritten as $(D^4 - D^3 - D^2 + D)y = t^2 + 4 + t\sin(t)$. First we solve $((D^4 - D^3 - D^2 + D)y = 0$. Solving the characteristic equation $r^4 - r^3 - r^2 + r = r^3(r-1) - r(r-1) = (r^3 - r)(r-1)$
$= r(r-1)(r+1)(r-1) = r(r-1)^2(r+1) = 0$, we have $r = 0$, 1, 1 and -1. The solutions of $(D^4 - D^3 - D^2 + D)y = 0$ are spanned by 1, e^t, te^t and e^{-t}.

Now we should find the annihilator of $t^2 + 4 + t\sin(t)$.

We have $D^3(t^2 + 4) = 0$, $(D^2 + 1)^2(t\sin(t)) = 0$.
So $D^3(D^2 + 1)^2(t^2 + 4 + t\sin(t)) = 0$.

The given equation is $(D^4 - D^3 - D^2 + D)y = t^2 + 4 + t\sin(t)$. Applying the annihilator $D^3(D^2 + 1)^2$ to the equation above, we have
$D^3(D^2+1)^2(D^4 - D^3 - D^2 + D)y = D^3(D^2+1)^2(t^2+4+t\sin(t)) = 0$. Solving the characteristic equation $r^4(r^2 + 1)^2(r^4 - r^3 - r^2 + r) = r^4(r^2 + 1)^2 r(r-1)^2(r+1) = r^5(r^2+1)^2(r-1)^2(r+1) = 0$, we have $r = 0$ with multiplicities 5 and $r = 1$ with multiplicities 2, r=-1 and $r = -i$ with multiplicities 2. The solution of $(D^4 - D^3 - D^2 + D)y = t^2 + 4 + t\sin(t)$ is spanned by 1, t, t^2, t^3, $\cos(t)$, $\sin(t)$, $t\cos(t)$, $t\sin(t)$, e^t, te^t and e^{-t}. Excluding those functions appeared as the solution of $(D^4 - D^3 - D^2 + D)y = 0$, we know that the particular solution is of the form $y_p(t) = c_1 t + c_2 t^2 + c_3 t^3 + c_4\cos(t) + c_5\sin(t) + c_6 t\cos(t) + c_7 t\sin(t)$.