Solution to Quiz #7 and HW 9

1. Quiz problem (Sec 4.3 Problem 18)

The given equation can be rewritten as

\[(D^4 + 2D^3 + 2D^2)y = 3e^t + 2te^{-t} + e^{-t}\sin(t).\]

First we solve \((D^4 + 2D^3 + 2D^2)y = 0\). Solving the characteristic equation \(r^4 + 2r^3 + 2r^2 - r^2(r^2 + 2r + 2) = r^2(r + 1)^2 + 1 = 0\), we have \(r = 0, 0, -1 \pm i\).

The solutions of \((D^4 + 2D^3 + 2D^2)y = 0\) are spanned by 1, \(t\), \(e^{-t}\cos(t)\) and \(e^{-t}\sin(t)\).

Now we should find the annihilator of \(3e^t + 2te^{-t} + e^{-t}\sin(t)\). We have \((D - 1)(e^t) = 0\), \((D + 1)^2(te^{-t}) = 0\) and \((D + 1)^2 + 1)(e^{-t}\sin(t)) = 0\). So \((D - 1)(D + 1)^2((D + 1)^2 + 1)(3e^t + 2te^{-t} + e^{-t}\sin(t)) = 0\).

The given equation is \((D^4 + 2D^3 + 2D^2)y = 3e^t + 2te^{-t} + e^{-t}\sin(t)\). Applying the annihilator \((D - 1)(D + 1)^2((D + 1)^2 + 1)\) to the equation above, we have

\[(D - 1)(D + 1)^2((D + 1)^2 + 1)(D^4 + 2D^3 + 2D^2)y \]
\[= (D - 1)(D + 1)^2((D + 1)^2 + 1)(3e^t + 2te^{-t} + e^{-t}\sin(t)) = 0.\]

Solving the characteristic equation \((r - 1)(r + 1)^2(((r + 1)^2 + 1)(r^2 + 2r^3 + 2r^2)\)
\[= (r - 1)(r + 1)^2(((r + 1)^2 + 1)r^2((r + 1)^2 + 1) = (r - 1)(r + 1)^2(((r + 1)^2 + 1)^2r^2 = 0,\]
we have \(r = 1, -1, -1, -1 \pm i, -1 \pm i, 0\) and 0. The solution of \((D^4 + 2D^3 + 2D^2)y = 3e^t + 2te^{-t} + e^{-t}\sin(t)\) is spanned by \(e^t\), \(e^{-t}\), \(t e^{-t}\cos(t)\), \(t e^{-t}\sin(t)\), \(te^{-t}\cos(t)\), \(te^{-t}\sin(t)\), \(1\) and \(t\). Excluding those functions appeared as the solution of \((D^4 + 2D^3 + 2D^2)y = 0\), we know that the particular solution is of the form \(y_p(t) = c_1e^t + c_2e^{-t} + c_3te^{-t} + c_4te^{-t}\cos(t) + c_5te^{-t}\sin(t).\)

2. (Sec 4.1 Problem 6) We rewrite the equation \((x^2 - 4)y''(t) + x^2y'' + 9y = 0\) as \(y''(t) + \frac{9}{x^2 - 4}y = 0\). Now the function \(\frac{x^2}{x^2 - 4}\) and \(\frac{9}{x^2 - 4}\) are continuous on \((-\infty, -2) \cup (-2, 2) \cup (2, \infty)\). So the solution exists on \((-\infty, -2) \cup (-2, 2) \cup (2, \infty)\).

3. (Sec 4.2 Problem 15) The characteristic equation of \(y^{(6)}(t) + y = 0\) is \(r^6 + 1 = 0\). So \(r^6 = -1 = e^{i(\pi + 2k\pi)}\) where \(k\) is an integer. Now \(r = e^{i(\pi/6 + k\pi)}\) for \(k = 0, 1, 2, 3, 4\) and 5. So \(r = e^{i(\pi/6)} = \cos(\pi/6) + i\sin(\pi/6) = \sqrt{3}/2 + i\frac{1}{2},\)
\(r = e^{i(\pi/2)} = e^{i(\pi/2)} = \cos(\pi/2) + i\sin(\pi/2) = i,\)
\(r = e^{i(2\pi/6)} = \cos(\pi/3) + i\sin(\pi/6) = -\sqrt{3}/2 + i\frac{1}{2},\)
\(r = e^{i(\pi/2)} = \cos(\pi/2) + i\sin(\pi/2) = -\sqrt{3}/2 - i\frac{1}{2},\)
\(r = e^{i(\pi/2)} = \cos(3\pi/6) + i\sin(3\pi/6) = -\sqrt{3}/2 - i\frac{1}{2}.\)

Thus the general solution is \(y(t) = c_1e^{\sqrt{3}/2} \cos(t/2) + c_2e^{i\sqrt{3}/2} \sin(t/2) + c_3 \cos(t) + c_4 \sin(t) + c_5 e^{-\sqrt{3}/2} \cos(t/2) + c_6 e^{-i\sqrt{3}/2} \sin(t/2).\)

MATH 3860: page 1 of 3
4. (Sec 4.3 Problem 1) The given equation can be rewritten as 
\[(D^3 - D^2 - D + 1)y = 2e^{-t} + 3.\]
First we solve \((D^3 - D^2 - D + 1)y = 0.\)
Solving the characteristic equation
\[r^3 - r^2 - r + 1 = r^2(r - 1) - (r - 1) = (r - 1)^2(r + 1) = 0,\]
we have \(r = 1, 1\) and \(-1.\) The solutions of \((D^3 - D^2 - D + 1)y = 0\) are spanned by \(e^t, te^t\) and \(e^{-t}.\)

Now we should find the annihilator of \(2e^{-t} + 3.\) We have \((D+1)(e^{-t}) = 0, D(3) = 0.\) So \((D+1)D(2e^{-t} + 3) = 0.\)

The given equation is \((D^3 - D^2 - D + 1)y = 2e^{-t} + 3.\) Applying the annihilator \((D+1)D\) to the equation above, we have \((D+1)D(D^3 - D^2 - D + 1)y = (D+1)D(2e^{-t} + 3) = 0.\) Solving the characteristic equation
\[(r-1)r(r^3 - r^2 - r + 1) = (r-1)2(r+1) = r(r-1)^2(r+1) = 0,\]
we have \(r = 0, 1, 1, 1\) and \(-1.\) The solution of \((D^3 - D^2 - D + 1)y = 2e^{-t} + 3\) is spanned by \(1, e^t, te^t, t^2e^t\) and \(e^{-t}.\) Excluding those functions appeared as the solution of \((D^3 - D^2 - D + 1)y = 0,\) we know that the particular solution is of the form \(y_p(t) = c_1e^t.\)

With \(y_p(t) = c_1e^t + c_2t^2e^t,\) we have \(y_p''(t) = 2c_2te^t + c_2t^2e^t, y_p'''(t) = 2c_2e^t + 2c_2te^t + 2c_2te^t + c_2t^2e^t = 2c_2e^t + 2c_2te^t + c_2t^2e^t\) and \(y_p''''(t) = 2c_2e^t + 4c_2e^t + 4c_2te^t + 2c_2te^t + 2c_2t^2e^t = 6c_2e^t + 6c_2te^t + c_2t^2e^t.\) So \(y_p''''(t) - (2c_2e^t + 4c_2te^t + c_2t^2e^t) - (2c_2e^t + 4c_2te^t + c_2t^2e^t) = c_1 + c_2t^2e^t = 2c_2e^t + c_1 = 2e^{-t} + 3\) if \(c_1 = 3\) and \(c_2 = 1.\) Thus the general solution of \((D^3 - D^2 - D + 1)y = 2e^{-t} + 3\) is \(y(t) = 3 + t^2e^{-t} + c_1e^t + c_2te^t + c_3e^{-t}.\)

5. (Sec 4.3 Problem 13)

The given equation can be rewritten as \((D^3 - 2D^2 + D)y = t^3 + 2e^t.\)
First we solve \((D^3 - 2D^2 + D)y = 0.\) Solving the characteristic equation
\[r^3 - 2r^2 + r = r(r^2 - 2r + 1) = r(r-1)^2 = 0,\]
we have \(r = 0, 1, 1\) and \(-1.\) The solutions of \((D^3 - 2D^2 + D)y = 0\) are spanned by \(1, e^t\) and \(te^t.\)

Now we should find the annihilator of \(t^3 + 2e^t.\)

We have \(D^4(t^3) = 0, (D-1)(e^t) = 0.\) So \(D^4(D-1)(t^3 + 2e^t) = 0.\)

The given equation is \((D^3 - 2D^2 + D)y = t^3 + 2e^t.\) Applying the annihilator \(D^4(D-1)\) to the equation above, we have \(D^4(D-1)(D^3 - 2D^2 + D)y = D^4(D-1)(t^3 + 2e^t) = 0.\)
Solving the characteristic equation
\[r^4(r-1)(r^3 - 2r^2 + r) = r^4(r-1)r(r-1)^2 = r^5(r-1)^3 = 0,\]
we have \(r = 0\) with multiplicities \(5\) and \(r = 1\) with multiplicities \(3.\) The solution of \((D^3 - 2D^2 + D)y = t^3 + 2e^t\) is spanned by \(1, t, t^2, t^3, t^4, e^t, te^t\) and \(t^2e^t.\) Excluding those functions appeared as the solution of \((D^3 - 2D^2 + D)y = 0,\) we know that the particular solution is of the form \(y_p(t) = c_1t + c_2t^2 + c_3t^3 + c_4t^4 + c_5t^2e^t.\)
The given equation can be rewritten as \((D^4 - D^3 - D^2 + D)y = t^2 + 4 + t\sin(t)\). First we solve \(((D^4 - D^3 - D^2 + D)y = 0\). Solving the characteristic equation \(r^4 - r^3 - r^2 + r = r^3(r - 1) - r(r - 1) = (r^3 - r)(r - 1) = r(r - 1)(r + 1)(r - 1) = r(r - 1)^2(r + 1) = 0\), we have \(r = 0, 1, 1\) and \(-1\). The solutions of \((D^4 - D^3 - D^2 + D)y = 0\) are spanned by \(1, e^t, te^t\) and \(e^{-t}\).

Now we should find the annihilator of \(t^2 + 4 + t\sin(t)\).

We have \(D^3(t^2 + 4) = 0, (D^2 + 1)^2(t\sin(t)) = 0\).

So \(D^3(D^2 + 1)^2(t^2 + 4 + t\sin(t)) = 0\).

The given equation is \((D^4 - D^3 - D^2 + D)y = t^2 + 4 + t\sin(t)\). Applying the annihilator \(D^3(D^2 + 1)^2\) to the equation above, we have \(D^3(D^2 + 1)^2(D^4 - D^3 - D^2 + D)y = D^3(D^2 + 1)^2(t^2 + 4 + t\sin(t)) = 0\). Solving the characteristic equation \(r^4(r^2 + 1)^2(r^4 - r^3 - r^2 + r) = r^4(r^2 + 1)^2r(r - 1)^2(r + 1) = r^5(r^2 + 1)^2(r - 1)^2(r + 1) = 0\), we have \(r = 0\) with multiplicities 5 and \(r = 1\) with multiplicities 2, \(r = -1\) and \(r = -i\) with multiplicities 2. The solution of \((D^4 - D^3 - D^2 + D)y = t^2 + 4 + t\sin(t)\) is spanned by \(1, t, t^2, t^3, \cos(t), \sin(t), t\cos(t), t\sin(t), e^t, te^t\) and \(e^{-t}\). Excluding those functions appeared as the solution of \((D^4 - D^3 - D^2 + D)y = 0\), we know that the particular solution is of the form \(y_p(t) = c_1t + c_2t^2 + c_3t^3 + c_4\cos(t) + c_5\sin(t) + c_6t\cos(t) + c_7t\sin(t)\).