Solution to Quiz #8 and HW 10

1. Quiz problem (Sec 6.2 Problem 18) Taking the Laplace's transform of the differential equation $y^{(4)} - y = 0$, we have $L(y^{(4)} - y) = 0$.
Using $L(y^{(4)}) = s^4L(y) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)$, we have
$s^4L(y) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - L(y) = 0$ and
$(s^4 - 1)L(y) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = 0$. Substituting $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 0$, we have $(s^4 - 1)L(y) - s^3 - s = 0$, $(s^4 - 1)L(y) = s^3 + s$. So we have $L(y) = \frac{s^3+s}{s^4-1} = \frac{s(s^2+1)}{(s^2-1)(s^2+1)} = \frac{s}{s^2-1}$ and $y(t) = L^{-1}(\frac{s}{s^2-1}) = \cosh(t)$.

2. (Sec 6.1 Problem a) Find the Laplace transform of $2t^2+\sin(2t)+e^t\cos(2t)$.
Recall that the $L(t^2) = \frac{2}{s^3}$, $L(\sin(2t)) = \frac{2}{s^2+2^2} = \frac{2}{s^2+4}$ and $L(e^t\cos(2t)) = \frac{s}{(s-1)^2+2^2} = \frac{s}{(s-1)^2+4}$. So $L(2t^2 + \sin(2t) + e^t\cos(2t)) = 2L(2t^2) + L(\sin(2t)) + L(e^t\cos(2t)) = \frac{4}{s^3} + \frac{2}{s^2+4} + \frac{3s}{(s-1)^2+4}$.

3. (Sec 6.1 Problem b) Find the Laplace transform of $te^{2t} - te^t sin(2t)$.
Recall that the $L(e^{2t}) = \frac{1}{s-2}$ and $L(e^t\sin(2t)) = \frac{2}{(s-1)^2+2^2} = \frac{2}{(s-1)^2+4}$. We have
$L(-te^{2t}) = (\frac{1}{s-1})' = (\frac{-1}{(s-1)^2})$ and $L(-te^t\sin(2t)) = (\frac{2}{(s-1)^2+4})' = \frac{-4(s-1)}{((s-1)^2+4)^2}$, i.e. $L(te^{2t}) = (\frac{1}{(s-1)^2})$ and $L(te^t\sin(2t)) = \frac{4(s-1)}{((s-1)^2+4)^2}$. So $L(te^{2t} - te^t\sin(2t)) = L(te^{2t}) - L(te^t\sin(2t)) = \frac{1}{(s-1)^2} - \frac{4(s-2)}{((s-1)^2+4)^2}$.

4. (Sec 6.2 Problem 12) Taking the Laplace's transform of the differential equation $y'' + 3y' + 2y = 0$, we have $L(y'' + 3y' + 2y) = L(0) = 0$. Using $L(y'') = s^2L(y) - sy(0) - y'(0)$ and $L(y') = sL(y) - y(0)$, we have $s^2L(y) - sy(0) - y'(0) + 3(sL(y) - y(0)) + 2L(y) = 0$ and
$(s^2+3s+2)L(y) - sy(0) - y'(0) - 3y(0) = 0$. Applying the initial conditions, we have $(s^2 + 3s + 2)L(y) = s + 3$, So we have $L(y(t)) = \frac{s+3}{(s^2+3s+2)}$. Using partial fractions, $\frac{s+3}{(s^2+3s+2)} = \frac{2}{s+1} - \frac{1}{s+2}$. ($\frac{s+3}{(s^2+3s+2)} = \frac{s+3}{(s+1)(s+2)} = \frac{a}{(s+1)} + \frac{b}{(s+2)}$. Multiplying $(s+1)(s+2)$, we have $s+3 = a(s+2)+b(s+1) = (a+b)s+2a+b$. Comparing the coefficient, we have $a + b = 1$ and $2a + b = 3$. This gives $a = 2$ and $b = -1$.)
Hence $y(t) = L^{-1}(\frac{2}{s+1} - \frac{1}{s+2}) = 2e^{-t} - e^{-2t}$.

5. (Sec 4.3 Problem 13)
Taking the Laplace's transform of the eq $y'' - 2y' + 2y = 0$,
we have $L(y'' - 2y' + 2y) = L(0) = 0$.
Using $L(y'') = s^2L(y) - sy(0) - y'(0)$ and $L(y') = sL(y) - y(0)$, we have
$s^2L(y) - sy(0) - y'(0) - 2(sL(y) - y(0)) + 2L(y) = 0$ and
$(s^2 - 2s + 2)L(y) - sy(0) - y'(0) + 2y(0) = 0$.

Applying the initial conditions,, we have
$(s^2-2s+2)L(y) = 1$ and $(s^2+4s+5)L(y) = 2s+3$. So we have $L(y) = \frac{1}{s^2-2s+2}$.

Note that $s^2 - 2s + 2 = (s-1)^2 + 1$. $y(t) = L^{-1}(\frac{1}{s^2-2s+2}) = L^{-1}(\frac{1}{s-1)^2+1}) = e^t \sin(t)$.