Solution to Quiz #9 and HW 11

1. Quiz problem (Sec 7.5 Problem 1)

We have

$$\text{Let} \quad A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}.$$

$$det(A-\lambda I) = det \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} = (3-\lambda)(-2-\lambda)+4 = \lambda^2-\lambda-2 = (\lambda-2)(\lambda+1).$$

Hence the eigenvalues of A are $\lambda = 2$ and $\lambda = -1$.

To find the eigenvector corresponding to $\lambda = 2$, we must solve $(A - \lambda I)v = 0$. Substituting A and $\lambda = 2$ gives

$$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This matrix equation is equivalent to equation $v_1 - 2v_2 = 0$ and $2v_1 - 4v_2 = 0$. From $v_1 - 2v_2 = 0$, we have $v_1 = 2v_2$ and

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2v_2 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

We may choose $v_2 = 1$ to get the eigenvector

$$v = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{with eigenvalue} \quad 2.$$

Now we want to find the eigenvector corresponding to $\lambda = -1$, we must solve $(A - \lambda I)v = 0$. Substituting A and $\lambda = -1$ gives

$$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This matrix equation is equivalent to equation $4v_1 - 2v_2 = 0$ and $2v_1 - v_2 = 0$. From $4v_1 - 2v_2 = 0$, we have $v_2 = 2v_1$ and

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We may choose $v_1 = 1$ to get the eigenvector

$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{with eigenvalue} \quad -1.$$

Thus the general solution is

$$x(t) = c_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2c_1 e^{2t} + c_2 c_2 e^{-t} \\ c_1 e^{2t} + 2c_2 c_2 e^{-t} \end{pmatrix}.$$

2. (Sec6.6 Problem 7)

$L(\int_0^t \sin(t-\tau)\cos(\tau)d\tau) = L((g*h)(t))$ where $g(t) = \sin(t)$ and $h(t) = \cos(t)$. So $L(\_0^t \sin(t-\tau)\cos(\tau)d\tau) = L(\sin(t))L(\cos(t)) = \frac{1}{s^2+1}\cdot\frac{s}{s^2+1} = \frac{s}{(s^2+1)^2}$.

3. (Sec6.6 Problem 14) $L(y''+2y'+2y) = L(sin(\alpha t))$
$\Rightarrow (s^2+2s+2)L(y) = L(sin(\alpha t))$ (Use $y(0) = y'(0) = 0$.)
$\Rightarrow L(y) = \frac{1}{s^2+2s+2}L(\sin(\alpha t))$ (Use $y(0) = y'(0) = 0$.)
So $y(t) = L^{-1}(\frac{1}{s^2+2s+2}L(\sin(\alpha t)))$. Let $f(t) = L^{-1}(\frac{1}{s^2+2s+2}) = L^{-1}(\frac{1}{(s+1)^2+1}) = e^{-t}\sin(t)$. and $g(t) = L^{-1}(L(\sin(\alpha t))) = \sin(\alpha t)$.
Then $y(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t e^{-(t-\tau)}\sin(t-\tau)\sin(\alpha\tau)d\tau$.

4. (Sec6.6 Problem 26) $L(\phi'(t) + \int_0^t (t-\xi)\phi(\xi)d\xi) = L(t)$.
$\Rightarrow L(\phi'(t)) + L(\int_0^t (t-\xi)\phi(\xi)d\xi) = \frac{1}{s^2}$
$\Rightarrow sL(\phi(t)) + L(t)L(\phi(t)) = \frac{1}{s^2}$
$\Rightarrow sL(\phi(t)) + \frac{1}{s^2}L(\phi(t)) = \frac{1}{s^2}$
$\Rightarrow (s+\frac{1}{s^2})L(\phi(t)) = \frac{1}{s^2}$
$\Rightarrow (\frac{s^3+1}{s^2})L(\phi(t)) = \frac{1}{s^2}$
$\Rightarrow L(\phi(t)) = \frac{1}{(s^3+1)}$
$\Rightarrow \phi(t) = L^{-1}(\frac{1}{s^3+1})$
Note that $s^3+1 = (s+1)(s^2-s+1)$ and $s^2-s+1 = (s-(1/2))^2+(3/4)$.
So $\frac{1}{s^3+1} = \frac{a}{s+1} + \frac{b(s-(1/2))+c}{s^2-s+1}$ and
$1 = a(s^2-s+1) + (b(s-(1/2))+c)(s+1)$
$= as^2 - as + a + bs^2 - (1/2)bs + cs + bs - (1/2)b + c$
$= (a+b)s^2 + (-a+(1/2)b+c)s + (a-(1/2)b+c)$.
Comparing the coefficient, we get $a+b=0$, $-a+(1/2)b+c=0$ and $a-(1/2)b+c=1$. Adding $-a+(1/2)b+c=0$ and $a-(1/2)b+c=1$, we get $2c=1$ and $c=(1/2)$. From $a+b=0$, we have $a=-b$. From $a=-b$ and $-a+(1/2)b+c=0$, we have $(3/2)b+(1/2)=0$ and $b=-1/3$. Using $a=-b$, we get $a=1/3$. So $\frac{1}{s^3+1} = 1/3\frac{1}{s+1} - (1/3)\frac{(s-(1/2))}{s^2-s+1} + (1/2)\frac{1}{s^2-s+1}$. Now $L^{-1}(\frac{(s-(1/2))}{s^2-s+1}) = L^{-1}(\frac{(s-(1/2))}{s-(1/2))^2+(\sqrt{3}/2)^2}) = e^{t/2}\cos(\sqrt{3}t/2)$ and $L^{-1}(\frac{1}{s^2-s+1}) = L^{-1}(\frac{1}{s-(1/2))^2+(\sqrt{3}/2)^2}) = \frac{2}{\sqrt{3}}e^{t/2}\sin(\sqrt{3}t/2)$.

Hence $y(t) = L^{-1}((1/3)\frac{1}{s+1} - (1/3)\frac{(s-(1/2))}{s^2-s+1} + (1/2)\frac{1}{s^2-s+1})$
$= (1/3)e^{-t} - (1/3)e^{t/2}\cos(\sqrt{3}t/2) + \frac{1}{\sqrt{3}}e^{t/2}\sin(\sqrt{3}t/2)$.

5. (Sec 7.5 Problem 5)

We have

$$\text{Let}\quad A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

$$det(A-\lambda I) = det\begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} = (-2-\lambda)^2 - 1 = \lambda^2+4\lambda+3 = (\lambda+3)(\lambda+1).$$

Hence the eigenvalues of A are $\lambda = -3$ and $\lambda = -1$.

To find the eigenvector corresponding to $\lambda = -3$, we must solve $(A-\lambda I)v = 0$. Substituting A and $\lambda = -3$ gives

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This matrix equation is equivalent to equation $v_1+v_2 = 0$ and $v_1+v_2 = 0$. From $v_1 + v_2 = 0$, we have $v_1 = -v_2$ and

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_2 \\ v_2 \end{pmatrix} = v_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

We may choose $v_2 = 1$ to get the eigenvector

$$v = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{with eigenvalue} \quad -3.$$

Now we want to find the eigenvector corresponding to $\lambda = -1$, we must solve $(A-\lambda I)v = 0$. Substituting A and $\lambda = -1$ gives

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This matrix equation is equivalent to equation $-v_1+v_2 = 0$ and $v_1-v_2 = 0$. From $v_1 - v_2 = 0$, we have $v_2 = v_1$ and

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We may choose $v_1 = 1$ to get the eigenvector

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{with eigenvalue} \quad 2.$$

Thus the general solution is

$$x(t) = c_1e^{-3t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -c_1e^{-3t} + c_2c_2e^{-t} \\ c_1e^{-3t} + c_2c_2e^{-t} \end{pmatrix}.$$

6. (Sec 7.5 Problem 6)

We have

$$\text{Let} \quad A = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}.$$

$$det(A-\lambda I) = det \begin{pmatrix} 5/4-\lambda & 3/4 \\ 3/4 & 5/4-\lambda \end{pmatrix} = (5/4-\lambda)^2-(9/16) = \lambda^2-(5/2)\lambda+1 = (\lambda-(1/2))(\lambda-2).$$

Hence the eigenvalues of A are $\lambda = 1/2$ and $\lambda = 2$.

To find the eigenvector corresponding to $\lambda = 1/2$, we must solve $(A-\lambda I)v = 0$. Substituting A and $\lambda = 1/2$ gives

$$\begin{pmatrix} 3/4 & 3/4 \\ 3/4 & 3/4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This matrix equation is equivalent to equation $(3/4)v_1+(3/4)v_2 = 0$ and $(3/4)v_1 + (3/4)v_2 = 0$. From $(3/4)v_1 + (3/4)v_2 = 0$, we have $v_1 = -v_2$ and

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_2 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

We may choose $v_2 = 1$ to get the eigenvector

$$v = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{with eigenvalue} \quad (1/2).$$

Now we want to find the eigenvector corresponding to $\lambda = 2$, we must solve $(A-\lambda I)v = 0$. Substituting A and $\lambda = 2$ gives

$$\begin{pmatrix} -3/4 & 3/4 \\ 3/4 & -3/4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This matrix equation is equivalent to equation $-(3/4)v_1 + (3/4)v_2 = 0$ and $(3/4)v_1 - (3/4)v_2 = 0$. From $(3/4)v_1 - (3/4)v_2 = 0$, we have $v_2 = v_1$ and

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We may choose $v_1 = 2$ to get the eigenvector

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{with eigenvalue} \quad 2.$$

Thus the general solution is

$$x(t) = c_1 e^{(1/2)t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -c_1 e^{(1/2)t} + c_2 c_2 e^{2t} \\ c_1 e^{(1/2)t} + c_2 c_2 e^{2t} \end{pmatrix}.$$