Solution to Review Problems for Midterm I

MATH 3860 – 002

Disclaimer: My solution is only correct up to a constant.
Please email(Mao-Pei.Tsui@math.utoledo.edu) me if you find any mistake.

(1) **Solving** $(2y - y^2)(1 + x^2y^2) = 0$, **we get** $2y - y^2 = y(2 - y) = 0$ **and** $y = 0$ **or** $y = 2$. **Note that** $1 + x^2y^2 > 0$.

We know that $y = 0$ **and** $y = 2$ **are solutions to** $\frac{dy}{dx} = (2y - y^2)(1 + x^2y^2)$.
Suppose $2 < y(0)$ **, we have** $y(x)$ **is decreasing to 2.**
Suppose $y(0) = 2$, **we have** $y(x) = 2$.
Suppose $0 < y(0) < 2$ **, we have** $y(x)$ **is increasing to 2.**
Suppose $y(0) = 0$, **we have** $y(x) = 0$.
Suppose $y(0) < 0$ **, we have** $y(x)$ **is decreasing to** $-\infty$.

(2) (a) **Let** $v = ax + by + c$. **Since** $\frac{dv}{dx} = a + b\frac{dy}{dx}$ **and** $\frac{dy}{dx} = F(ax + by + c)$, **we have** $\frac{dv}{dx} = a + bF(ax + by + c) = a + bF(v)$. **Thus** $\int \frac{dv}{a+bF(v)} = \int dx$.

(b) **Let** $v = x + y + 1$. **Since** $\frac{dv}{dx} = 1 + \frac{dy}{dx}$ **and** $\frac{dy}{dx} = (x + y + 1)^2 = v^2$, **we have** $\frac{dv}{dx} = 1 + v^2$. **Thus** $\int \frac{dv}{1+v^2} = \int dx$, $\arctan(v) = x + c$ **and** $v = \tan(x + c)$. **Recall that** $v = x + y + 1$, **we have** $y = v - x - 1 = \tan(x + c) - x - 1$.

(3) (a) **Let** $v = \frac{y}{x}$, **i.e.** $y = xv$. **Using** $\frac{dy}{dx} = v + x\frac{dv}{dx}$ **and** $\frac{dy}{dx} = F(\frac{y}{x}) = F(v)$, **we have** $v + x\frac{dv}{dx} = F(v)$. **It can be rewritten as** $x\frac{dv}{dx} = F(v) - v$ **which can be solved by** $\int \frac{dv}{F(v)-v} = \int \frac{dx}{x}$.

(b) **The equation** $x^2\frac{dy}{dx} = y^2 + xy - x^2$ **can be simplified as** $\frac{dy}{dx} = \frac{y^2+xy-x^2}{x^2} = (\frac{y}{x})^2 + \frac{y}{x} - 1$. **Let** $v = \frac{y}{x}$. **Note that** $(\frac{y}{x})^2 + \frac{y}{x} - 1 = v^2 + v - 1$. **Following the same computation as above, we have** $x\frac{dv}{dx} = v^2 - 1$. **(Using** $y = vx$ **and** $\frac{dy}{dx} = v + x\frac{dv}{dx}$, **we get** $v + x\frac{dv}{dx} = v^2 + v - 1$.)
It can be solved by $\int \frac{dv}{v^2-1} = \int \frac{dx}{x}$. **Note that** $\int \frac{dv}{v^2-1} = \int \frac{1}{2(v-1)} - \frac{1}{2(v+1)}dv = \frac{1}{2}\ln|\frac{v-1}{v+1}| + D$. **Hence** $\frac{1}{2}\ln|\frac{v-1}{v+1}| = \ln|x| + C$ **and** $\ln|\frac{v-1}{v+1}| = 2\ln|x| + c = \ln|x|^2 + c$. **Therefore** $\frac{v-1}{v+1} = Cx^2$ **and** $v = \frac{1+Cx^2}{1-Cx^2}$. **Recall that** $v = \frac{y}{x}$. **It follows that** $y = xv = x \cdot \frac{1+Cx^2}{1-Cx^2}$.

(4) (a) **Separate the equation, we have** $\int(3y^2 - 6y)dy = \int 2xdx$ **and** $y^3 - 3y^2 = x^2 + C$. **Plugging** $y(0) = 1$, **we have** $C = -2$ **Thus the solution** (x, y) **satisfies** $y^3 - 3y^2 = x^2 - 2$. **From the equation** $y' = \frac{2x}{3y^2-6y}$, **we know that the solution** y **will not exist if** $3y^2 - 6y = 3y(y - 2) = 0$, **that is** $y = 0$ **and** $y = 2$. **If** $y = 0$, **we have** $0 = x^2 - 2$, **i.e.** $x = \pm\sqrt{2}$. **From the**

graph in Figure 3, we know that the solution exists on the interval $-\sqrt{2} < x < \sqrt{2}$.

(b) Using $y^3 - 3y^2 = x^2 + C$ and $y(\sqrt{18}) = 4$, we have $64 - 48 = 18 + C$ and $C = -2$. So the solution (x, y) satisfies $y^3 - 3y^2 = x^2 - 2$. From the graph of $y^3 - 3y^2 = x^2 - 2$, we know that the solution exists on the interval $-\infty < x < \infty$.

(5) (a) $y(t) = \frac{2t}{3} + \frac{1}{9} + Ce^{-3t}$.Thus $\lim_{t\to\infty} y(t) - \frac{2t}{3} - \frac{1}{9} = 0$.
$(\mu(t) = e^{\int 3dt} = e^{3t}$. $y(t) = \frac{\int \mu(t)(2t+1)dt}{\mu(t)} = \frac{\int e^{3t}(2t+1)dt}{e^{3t}} = \frac{\frac{2t}{3}e^{3t} + \frac{1}{9}e^{3t} + C}{e^{3t}} = \frac{2t}{3} + \frac{1}{9} + Ce^{-3t}$.)

(b) $y(t) = \frac{1}{4}e^{-t} + (y_0 - \frac{1}{4})e^{3t}$. Note that $\lim_{t\to\infty} e^{3t} = \infty$ and $\lim_{t\to\infty} e^{-t} = 0$. The condition $y_0 < \frac{1}{4}$ will imply that $\lim_{t\to\infty} y(t) = -\infty$.
$(\mu(t) = e^{\int -3dt} = e^{-3t}$. $y(t) = \frac{\int \mu(t)(-e^{-t})dt}{\mu(t)} = \frac{\int e^{-3t}(-e^{-t})dt}{e^{-3t}} = \frac{\frac{1}{4}e^{-4t} + C}{e^{-3t}} = \frac{1}{4}e^{-t} + Ce^{3t}$. Use $y(0) = y_0$. We have $C = y_0 - \frac{1}{4}$.)

(c) $y(t) = t^3 \ln|t| + Ct^3$.
(Rewrite $t^3 + 3y - t\frac{dy}{dt} = 0$ as $-t\frac{dy}{dt} + 3y = -t^3$.
Dividing the equation by $-t$, we get $y' - \frac{3}{t}y = t^2$.
We have $p(t) = -\frac{3}{t}$ and $g(t) = t^2$.
The integrating factor is $\mu(t) = e^{\int p(t)dt} = e^{\int -\frac{3}{t}dt} = e^{-3\ln t} = e^{\ln t^{-3}} = t^{-3}$.
So $y(t) = \frac{\int \mu(t)g(t)dt}{\mu(t)} = \frac{\int t^{-3}t^2 dt}{t^{-3}} = \frac{\int t^{-1}dt}{t^{-3}} = \frac{\ln|t| + C}{t^{-3}} = t^3 \ln|t| + Ct^3$.)

(d) $y(t) = \frac{(t^2+t)}{\sqrt{2t+1}} + \frac{C}{\sqrt{2t+1}}$.
(Dividing the equation by $2t + 1$, we get $y' + \frac{1}{2t+1}y = (2t + 1)^{\frac{1}{2}}$. The integrating factor is $\mu(t) = e^{\int \frac{1}{2t+1}dt} = e^{\frac{1}{2}\ln|2t+1|} = e^{\ln|2t+1|^{\frac{1}{2}}} = |2t + 1|^{\frac{1}{2}}$. So $y(t) = \frac{\int |2t+1|^{\frac{1}{2}} \cdot (2t+1)^{\frac{1}{2}} dt}{|2t+1|^{\frac{1}{2}}} = \frac{\int (2t+1)dt}{(2t+1)^{\frac{1}{2}}} = \frac{t^2+t+C}{(2t+1)^{\frac{1}{2}}} = \frac{(t^2+t)}{\sqrt{2t+1}} + \frac{C}{\sqrt{2t+1}}$.)

(e) Rewrite the equation $t\frac{dy}{dt} = 6y + 12t^4y^{\frac{2}{3}}$ as $y' = \frac{6}{t}y + 12t^3y^{\frac{2}{3}} = 0$.
Let $v = y^{1-\frac{2}{3}} = y^{\frac{1}{3}}$. Then $\frac{dv}{dt} = \frac{1}{3}y^{-\frac{2}{3}}\frac{dy}{dt} = \frac{1}{3}y^{-\frac{2}{3}}(\frac{6}{t}y + 12t^3y^{\frac{2}{3}}) = \frac{2}{t}y^{\frac{1}{3}} + 4t^3 = \frac{2}{t}v + 4t^3$.
So $\frac{dv}{dt} - \frac{2}{t}v = 4t^3$. The integrating factor is $\mu(t) = e^{\int -\frac{2}{t}dt} = e^{-2\ln t} = t^{-2}$. The solution of $\frac{dv}{dt} - \frac{2}{t}v = 4t^3$ is $v(t) = \frac{\int \mu(t)4t^3}{\mu(t)} = \frac{\int t^{-2} \cdot 4t^3 dt}{t^{-2}} = \frac{\int 4tdt}{t^{-2}} = \frac{2t^2+C}{t^{-2}} = 2t^4 + Ct^2$. Recall that $v = y^{\frac{1}{3}}$. We have $y = v^3$ and $y = v^3 = (2t^4 + Ct^2)^3$.

(f) This can be rewritten as $\frac{dy}{dx} = 5x^4y^2 - 4x^2y^2 = y^2(5x^4 - 4x^2)$. The general solution is $y = \frac{-1}{x^5 - \frac{4}{3}x^3 + C}$.

(g) Let $M = 6xy^3 + 2y^4$ and $N = 9x^2y^2 + 8xy^3 + y^3$. Note that $M_y = 18xy^2 + 8y^3$ and $N_x = 18xy^2 + 8y^3$. Now $M_y = N_x$. This equation is exact. Solving $F_x = 6xy^3 + 2y^4$, we have $F = \int(6xy^3 + 2y^4)dx = 3x^2y^3 + 2xy^4 + g(y)$. Using $F_y = N = 9x^2y^2 + 8xy^3 + y^3$, we get $(3x^2y^3 + 2xy^4 + g(y))_y = 9x^2y^2 + 8xy^3 + y^3$

and $9x^2y^2 + 8xy^3 + g'(y) = 9x^2y^2 + 8xy^3 + y^3$. So $g'(y) = y^3$ and $g(y) = \int y^3 dy = \frac{y^4}{4}$. Hence $F(x,y) = 3x^2y^3 + 2xy^4 + \frac{y^4}{4}$ and the solution satisfies $F(x,y) = 3x^2y^3 + 2xy^4 + \frac{y^4}{4} = C$.

(h) $e^y + y\cos(x) + (xe^y + \sin(x) + e^y)\frac{dy}{dx} = 0$
Let $M = e^y + y\cos(x)$ and $N = xe^y + \sin(x) + e^y$. Note that $M_y = e^y + \cos(x)$ and $N_x = e^y + \cos(x)$. Now $M_y = N_x$. This equation is exact. Solving $F_x = e^y + y\cos(x)$, we have $F = \int(e^y + y\cos(x))dx = xe^y + y\sin(x) + g(y)$. Using $F_y = N = xe^y + \sin(x) + e^y$, we get $(xe^y + y\sin(x) + g(y))_y = xe^y + \sin(x) + e^y$ and $xe^y + \sin(x) + g'(y) = xe^y + \sin(x) + e^y$. So $g'(y) = e^y$ and $g(y) = \int e^y dy = e^y$. Hence $F(x,y) = xe^y + \sin(x)y + e^y$ and the solution satisfies $F(x,y) = xe^y + \sin(x)y + e^y = C$.

(i) Rewrite the equation $2x^2y - x^3\frac{dy}{dx} = y^3$ as $y' - \frac{2}{x}y = -\frac{y^3}{x^3}$.
Let $v = y^{1-3} = y^{-2}$. Then $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx} = -2y^{-3}(\frac{2}{x}y - \frac{y^3}{x^3}) = -\frac{4}{x}y^{-2} + \frac{2}{x^3} = -\frac{4}{x}v + \frac{2}{x^3}$.
So $\frac{dv}{dx} + \frac{4}{x}v = \frac{2}{x^3}$. The integrating factor is $\mu(x) = e^{\int \frac{4}{x}dx} = e^{4\ln x} = x^4$. The solution of $\frac{dv}{dx} + \frac{4}{x}v = \frac{2}{x^3}$ is $v(x) = \frac{\int \mu(x)\frac{2}{x^3}}{\mu(x)} = \frac{\int x^4 \cdot \frac{2}{x^3}dx}{x^4} = \frac{\int 2xdx}{x^4} = \frac{x^2+C}{x^4} = x^{-2} + Cx^{-4}$. Recall that $v = y^{-2}$. We have $y = \pm v^{-\frac{1}{2}}$ and $y = \pm(x^{-2} + Cx^{-4})^{-\frac{1}{2}}$.

(6) Obviously, $y_1(t) = 0$ is a solution of $\frac{dy}{dt} = \frac{y\sin(x)}{1+t^2+y^2}$. Now we have $y(0) = -2 < 0 = y_1(0)$. Obviously, the solution to $\frac{dy}{dt} = \frac{y\sin(x)}{1+t^2+y^2}$ is unique. We have $y(t) < y_1(0) = 0$, i.e. y is always negative.

(7) The equation $(9 - t^2)\frac{dy}{dt} + \frac{y}{t} = \cos(t)$ can be rewritten as $\frac{dy}{dt} + \frac{1}{t(3-t)(3+t)}y = \frac{\cos(t)}{(3-t)(3+t)}$. Both $\frac{1}{t(3-t)(3+t)}$ and $\frac{\cos(t)}{(3-t)(3+t)}$ are continuous on $(-\infty,-3) \cup (-3,0) \cup (0,3) \cup (0,\infty)$.
- **(a) Since $-1 \in (-3,0)$, this solution exists on the interval $(-3,0)$.**
- **(b) Since $1 \in (0,3)$, this solution exists on the interval $(0,3)$.**
- **(c) Since $4 \in (3,\infty)$, this solution exists on the interval $(3,\infty)$.**
- **(d) Since $-4 \in (-\infty,-3)$, this solution exists on the interval $(-\infty,-3)$.**

(8) In each problem, determine the equilibrium points, and classify each one as asymptotically stable, unstable, or semistable.
- **(a) The equilibrium points are $y = k\pi$ where k is an integer. Since $y^2\sin^2(y) > 0$ if $y \neq k\pi$, we know that all the equilibrium points are semistable.**
- **(b) The equilibrium points are $y = k\pi$ where k is an integer. The sign graph of $y\sin(y)$ and $\sin(y)$ is the same when $y > 0$. The sign graph of $y\sin(y)$ and $-\sin(y)$ is the same when $y < 0$. Thus $y\sin(y) > 0$ when $2k\pi < y < (2k+1)\pi$ or $-(2k+1)\pi < y < -2k\pi$ where k is**

an nonnegative integer. Thus $y\sin(y) < 0$ **when** $(2k+1)\pi < y < (2k+2)\pi$ **or** $-(2k+2)\pi < y < -(2k+1)\pi$ **where** k **is an nonnegative integer. Thus** 0 **is an semistable equilibrium point.** $\{\pi, 3\pi, 5\pi, \cdots\}$ **and** $\{-2\pi, -4\pi, -6\pi, \cdots\}$ **are asymptotically stable equilibrium points.** $\{2\pi, 4\pi, 6\pi, \cdots\}$ **and** $\{-\pi, -3\pi, -5\pi, \cdots\}$ **are unstable equilibrium points.**

(c) Let $f(y) = (-y^3+3y^2-2y)(y-3)^2$**. We have** $f(y) = -y(y^2-3y+2)(y-3)^2 = -y(y-1)(y-2)(y-3)^2$**. Thus** $f(y) > 0$ **when** $y \in (-\infty, 0) \cup (1, 2)$ **and** $f(y) < 0$ **when** $y \in (0, 1) \cup (2, 3) \cup (3, \infty)$**. Therefore** $\{0, 2\}$ **are are asymptotically stable equilibrium points, 1 is an unstable equilibrium point and 3 is a semistable equilibrium point.**

(d) Let $f(y) = y^3 - 3y^2 + 2y = y(y-1)(y-2)$**. Thus** $f(y) > 0$ **when** $y \in (0, 1) \cup (2, \infty)$ **and** $f(y) < 0$ $y \in (-\infty, 0) \cup (1, 2)$**. Hence** $y = 1$ **are are asymptotically stable equilibrium points,** $y = 0$ **and** $y = 2$ **are unstable equilibrium points.**