Solution to Review Problems for Midterm II

MATH 3860

(1) (a) Rewrite the equation
$$(t^2-16)y''(t)+ty'(t)+\frac{9}{t-3}y=0$$
as $y''+\frac{t}{t^2-16}y'+\frac{9}{(t^2-16)(t-3)}y=0$. So $p(t)=\frac{t}{t^2-16}$ and $q(t)=\frac{9}{(t^2-16)(t-3)}$. Hence $p(t)$ is continuous if $t\in(-\infty,-4)\cup(4,\infty)$ and $q(t)$ is continuous if $t\in(-\infty,-4)\cup(-4,3)\cup(3,\infty)$. Therefore $p(t)$ and $q(t)$ are continuous if $t\in(-\infty,-4)\cup(-4,3)\cup(3,\infty)$. The initial conditions are $y(2)=1$ and $y'(2)=2$. We have $2\in(-4,3)$. Thus the solution exists on the interval $(-4,3)$.

(b) The initial conditions are $y(-1)=1$ and $y'(-1)=2$. We have $-1\in(-4,3)$. Thus the solution exists on the interval $(-4,3)$.

(2) (a) Rewrite the equation $x^2y''(x)+xy'(x)+x^2y(x)=0$ as $y''+\frac{1}{x}y'+y=0$. Let $p(x)=\frac{1}{x}$. Now the Wronskian is $W(x)=ce^{-\int p(x)dx}=ce^{-\int(\frac{1}{x})dx}=ce^{-\ln x}=\frac{c}{x}$.

(b) Now the Wronskian is $W(y_1,y_2)(x)=\frac{c}{x}$. Using $W(y_1,y_2)(1)=5$, we have $W(y_1,y_2)(1)=c$. So $c=5$ and $W(y_1,y_2)(x)=\frac{5}{x}$.

(3) Find the general solution of the following differential equations.

(a) $y''(t)+6y'(t)+9y=0$. The characteristic equation of $y''(t)+6y'(t)+9y(t)=0$ is $r^2+6r+9=(r+3)^2=0$. We have repeated roots $r=-3$. Thus the general solution is $y(t)=c_1e^{-3t}+c_2te^{-3t}$.

(b) $y''(t)+5y'(t)+4y=0$. The characteristic equation of $y''(t)+5y'(t)+4y=0$ is $r^2+5r+4=(r+1)(r+4)=0$. We have $r=-1$ or $r=-4$. Thus the general solution is $y(t)=c_1e^{-t}+c_2e^{-4t}$.

(c) $y''(t)+4y'(t)+5y=0$. The characteristic equation of $y''(t)+4y'(t)+5y=0$ is $r^2+4r+5=0$. We have $r=-2\pm i$. Note that $e^{(-2+i)t}=e^{-2t}e^{it}=e^{-2t}\cos(t)+ie^{-2t}\sin(t)$. Thus the general solution is $y(t)=c_1e^{-2t}\cos(t)+c_2e^{-2t}\sin(t)$.

(d) $t^2y''(t)+7ty'(t)+8y(t)=0$. Suppose $y(t)=t^r$, we have $y'(t)=rt^{r-1}$ and $y''(t)=r(r-1)t^{r-2}$. Thus $t^2y''(t)+7ty'(t)+8y(t)=(r(r-1)+7r+8)t^r=(r^2+6r+8)t^r$. Thus $y=t^r$ is a solution of $t^2y''(t)+7ty'(t)+8y(t)=0$ if $r^2+6r+8=(r+2)(r+4)=0$. The roots of $r^2+6r+8=0$ are -2 and -4. Therefore the general solution is $y(t)=c_1t^{-2}+c_2t^{-4}$.

(e) $t^2y''(t)+7ty'(t)+10y(t)=0$. Suppose $y(t)=t^r$, we have $y'(t)=rt^{r-1}$ and $y''(t)=r(r-1)t^{r-2}$. Thus $t^2y''(t)+7ty'(t)+10y(t)=(r(r-1)+7r+10)t^r=(r^2+6r+10)t^r$. Thus $y=t^r$ is a solution of $t^2y''(t)+7ty'(t)+10y(t)=0$ if $r^2+6r+10=0$. The roots of $r^2+6r+10=0$ are $-3+i$ and $-3-i$. Note that $t=e^{\ln t}$ and $t^{-3+i}=t^{-3}e^{i\ln t}=t^{-3}\cos(\ln t)+it^{-3}\sin(\ln t)$. Therefore the general solution is $c_1t^{-3}\cos(\ln t)+c_2t^{-3}\sin(\ln t)$.

(f) $t^2y''(t) + 5ty'(t) + 4y(t) = 0$. Suppose $y(t) = t^r$, we have $y'(t) = rt^{r-1}$ and $y''(t) = r(r-1)t^{r-2}$. Thus $t^2y''(t) + 5ty'(t) + 4y(t) = (r(r-1) + 5r + 4)t^r = (r^2 + 4r + 4)t^r$. Thus $y = t^r$ is a solution of $t^2y''(t) + 5ty'(t) + 4y(t) = 0$ if $r^2 + 4r + 4 = (r+2)^2 = 0$. The roots of $r^2 + 4r + 4 = 0$ are -2. Therefore the general solution is $c_1t^{-2} + c_2t^{-2}\ln t)$.

(g) $t^2y''(t) + ty'(t) + 9y = 0$. Suppose $y(t) = t^r$, we have $y'(t) = rt^{r-1}$ and $y''(t) = r(r-1)t^{r-2}$. Thus $t^2y''(t) + ty'(t) + 9y(t) = (r(r-1) + r + 9)t^r = (r^2 + 9)t^r$. Thus $y = t^r$ is a solution of $t^2y''(t) + \alpha ty'(t) + \beta y(t) = 0$ if $r^2 + 9 = 0$. The roots of $r^2 + 9 = 0$ are $3i$ and $-3i$. Note that $t = e^{\ln t}$ and $t^{3i} = e^{i3\ln t} = \cos(3\ln t) + i\sin(3\ln t)$. Therefore the general solution is $c_1\cos(3\ln t) + c_2\sin(3\ln t)$.

(4) Find the solution of the following initial value problems.

(a) $y''(t) + 4y'(t) + 5y = 0$, $y(0) = 1$ and $y'(0) = 3$. From (1c), we have $y(t) = c_1e^{-2t}\cos(t) + c_2e^{-2t}\sin(t)$ and $y'(t) = -2c_1e^{-2t}\cos(t) - c_1e^{-2t}\sin(t) - 2c_2e^{-2t}\sin(t) + c_2e^{-2t}\cos(t) = (-2c_1 + c_2)e^{-2t}\cos(t) + (-c_1 - 2c_2)e^{-2t}\sin(t)$. Using $y(0) = 1$ and $y'(0) = 3$, we have $c_1 = 1$ and $-2c_1 + c_2 = 3$. So $c_1 = 1$ and $c_2 = 5$. Hence $y(t) = e^{-2t}\cos(t) + 5e^{-2t}\sin(t)$.

(b) $t^2y''(t) + 7ty'(t) + 10y(t) = 0$, $y(1) = 2$ and $y'(1) = -5$. From (1e), we have $y(t) = c_1t^{-3}\cos(\ln t) + c_2t^{-3}\sin(\ln t)$ and $y'(t) = -3c_1t^{-4}\cos(\ln t) - c_1t^{-3}\frac{\sin(\ln t)}{t} - 3c_2t^{-4}\sin(\ln t) + c_2t^{-3}\frac{\cos(\ln t)}{t} = (-3c_1 + c_2)t^{-4}\cos(\ln t) + (-c_1 - 3c_2)t^{-4}\sin(\ln t)$. Using $y(1) = 2$ and $y'(1) = -5.$, we have $c_1 = 2$ and $-3c_1 + c_2 = -5$. So $c_1 = 2$ and $c_2 = 1$. Hence $y(t) = 2t^{-3}\cos(\ln t) + t^{-3}\sin(\ln t)$.

(5) In the following problems, a differential and one solution y_1 are given. Use the method of reduction of order to find the general solution solution.

(a) $t^2y''(t) - t(t+2)y'(t) + (t+2)y(t) = 0; y_1(t) = t$. Rewrite the equation $t^2y''(t) - t(t+2)y'(t) + (t+2)y(t) = 0$ as $y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 0$. So $p(t) = -\frac{t+2}{t}$. Let y be a solution of $t^2y''(t) - t(t+2)y'(t) + (t+2)y(t) = 0$. We have $(\frac{y}{y_1})' = \frac{y_1y' - y_1'y}{y_1^2} = \frac{W(t)}{y_1^2} = \frac{Ce^{-\int p(t)dt}}{t^2} = \frac{Ce^{\int \frac{t+2}{t}dt}}{t^2} = \frac{Ce^{\int(1+\frac{2}{t})dt}}{t^2} = \frac{Ce^{(t+2\ln(t))}}{t^2} = \frac{Ce^te^{2\ln t}}{t^2} = \frac{Ce^tt^2}{t^2} = Ce^t$. So $\frac{y}{y_1} = \int Ce^tdt = Ce^t + D$ and $y = y_1(Ce^t + D) = t(Ce^t + D) = Cte^t + Dt$. So the general solution is $y = Cte^t + Dt$.

(b) $(t+1)y''(t) - (t+2)y'(t) + y(t) = 0; y_1(t) = e^t$. Rewrite the equation $(t+1)y''(t) - (t+2)y'(t) + y(t) = 0$ as $y'' - \frac{t+2}{t+1}y' + \frac{1}{t+1}y = 0$. So $p(t) = -\frac{t+2}{t+1}$. Let y be a solution of $(t+1)y''(t) - (t+2)y'(t) + y(t) = 0$. We have $(\frac{y}{y_1})' = \frac{y_1y' - y_1'y}{y_1^2} = \frac{W(t)}{y_1^2} = fracCe^{-\int p(t)dt}e^{2t} = \frac{Ce^{\int \frac{t+2}{t+1}dt}}{t^2} = \frac{Ce^{\int(1+\frac{1}{t+1})dt}}{e^{2t}} = \frac{Ce^{(t+\ln(t+1))}}{e^{2t}} = \frac{Ce^te^{\ln(t+1)}}{e^{2t}} = \frac{Ce^t(t+1)}{e^{2t}} = Ce^{-t}(t+1) = C(te^{-t} + e^{-t})$. So $\frac{y}{y_1} = \int C(te^{-t} + e^{-t})dt =$

$C(-te^{-t}-2e^{-t})+D$ and $y = y_1(C(-te^{-t}-2e^{-t})+D) = e^t(C(-te^{-t}-2e^{-t})+D) = -C(t+2)+De^t$. So the general solution is $y = c(t+2)+de^t$.

(6) Find the general solution of the following differential equations.

(a) $y''(t)+5y'(t)+6y(t) = e^t+\sin(t)$.
Solving $r^2+5r+6 = (r+2)(r+3) = 0$, we know that the solution of $y''(t)+5y'(t)+6y(t) = 0$ is $y(t) = c_1e^{-2t}+c_2e^{-3t}$. We try $y_p = ce^t+d\sin(t)+e\cos(t)$ to be a particular solution of $y''(t)+5y'(t)+6y(t) = e^t+\sin(t)$. We have $y_p = ce^t+d\sin(t)+e\cos(t)$, $y_p' = ce^t+d\cos(t)-e\sin(t)$, $y_p'' = ce^t-d\sin(t)-e\cos(t)$ and $y_p''(t)+5y_p'(t)+6y_p(t) = (ce^t-d\sin(t)-e\cos(t))+5(ce^t+d\cos(t)-e\sin(t))+6(ce^t+d\sin(t)+e\cos(t)) = (c+5c+6c)e^t+(-d-5e+6d)\sin(t)+(-e+5d+6e)\cos(t) = 12ce^t+(5d-5e)\sin(t)+(5d+5e)\cos(t) = e^t+\sin(t)$ if $12c = 1$, $5d-5e = 1$ and $5d+5e = 0$. So $c = \frac{1}{12}$, $d = \frac{1}{10}$ and $e = -\frac{1}{10}$. Thus the general solution of $y''(t)+5y'(t)+6y(t) = e^t+\sin(t)$ is $y(t) = \frac{1}{12}e^t+\frac{1}{10}\sin(t)+\frac{1}{10}\cos(t)+c_1e^{-2t}+c_2e^{-3t}$.

(b) $y''(t)+4y = 2\sin(2t)+3\cos(t)$ Solving $r^2+4 = 0$, we know that the solution of $y''(t)+4y = 0$ is $y(t) = c_1\sin(2t)+c_2\cos(2t)$. We try $y_p = ct\sin(2t)+dt\cos(2t)+e\sin(t)+f\cos(t)$ to be a particular solution of $y''(t)+4y = 2\sin(2t)+3\cos(t)$. We have $y_p = ct\sin(2t)+dt\cos(2t)+e\sin(t)+f\cos(t)$,
$y_p' = c\sin(2t)+2ct\cos(2t)+d\cos(2t)-2dt\sin(2t)+e\cos(t)-f\sin(t)$,
$y_p'' = 4c\cos(2t)-4ct\sin(2t)-4d\sin(2t)-4dt\cos(2t)-e\sin(t)-f\cos(t)$ and
$y_p''(t)+4y_p(t) = 4c\cos(2t)-4d\sin(2t)+3e\sin(t)+3f\cos(t) = 2\sin(2t)+3\cos(t)$ if $c = 0$, $d = -\frac{1}{2}$, $e = 0$ and $f = 1$. Thus the general solution of $y''(t)+4y = 2\sin(2t)+3\cos(t)$ is $y(t) = -\frac{1}{2}t\cos(2t)+\cos(t)+c_1\sin(2t)+c_2\cos(2t)$.

(c) $y''(t)+4y = 4e^{4t}$ We try $y_p(t) = ce^{4t}$. Then $y_p' = 4ce^{4t}$, $y_p'' = 16ce^{4t}$. So $y_p''(t)+4y_p = 20ce^{4t} = 4e^{4t}$ if $c = \frac{1}{5}$. The general solution is $y(t) = \frac{1}{5}e^{4t}+c_1\sin(2t)+c_2\cos(2t)$.

(d) $y''(t)+4y'+4y(t) = e^{-2t}+e^{2t}$ Solving $r^2+4r+4 = (r+2)^2 = 0$, we know that the solution of $y''(t)+4y+4y(t) = 0$ is $y(t) = c_1e^{-2t}+c_2te^{-2t}$. We try $y_p = ct^2e^{-2t}+de^{2t}$ to be a particular solution of $y''(t)+4y+4y(t) = e^{-2t}+e^{2t}$. We have $y_p = ct^2e^{-2t}+de^{2t}$, $y_p' = 2cte^{-2t}-2ct^2e^{-2t}+2de^{2t}$, $y_p'' = 2ce^{-2t}-4cte^{-2t}-4cte^{-2t}+4ct^2e^{-2t}+4de^{2t} = 2ce^{-2t}-8cte^{-2t}+4ct^2e^{-2t}+4de^{2t}$ and $y_p''(t)+4y_p'(t)+4y_p(t) = (2ce^{-2t}-8cte^{-2t}+4ct^2e^{-2t}+4de^{2t})+4(2cte^{-2t}-2ct^2e^{-2t}+2de^{2t})+4(ct^2e^{-2t}+de^{2t}) = 2ce^{-2t}+16de^{2t}$. So $y_p''(t)+4y_p'(t)+4y_p(t) = e^{-2t}+e^{2t}$ if $2c = 1$ and $16d = 1$, $c = \frac{1}{2}$ and $d = \frac{1}{16}$. Thus the general solution of $y''(t)+4y'+4y(t) = e^{-2t}+e^{2t}$ is $y(t) = \frac{1}{2}t^2e^{-2t}+\frac{1}{16}e^{2t}+c_1e^{-2t}+c_2te^{-2t}$.

(e) $y''(t)+5y'(t)+6y(t) = t^2+1$. Solving $r^2+5r+6 = (r+2)(r+3) = 0$, we know that the solution of $y''(t)+5y+6y(t) = 0$ is $y(t) = c_1e^{-2t}+c_2e^{-3t}$. We try

$y_p(t) = at^2+bt+c$ to be a particular solution of $y''(t)+5y'(t)+6y(t) = t^2+1$. So $y_p' = 2at+b$, $y_p'' = 2a$ and $y_p''(t)+5y_p'(t)+6y_p(t) = 2a+5(2at+b)+6(at^2+bt+c) = 6at^2 + (10a + 6b)t + 2a + 5b + 6c$. So $y_p''(t) + 5y_p'(t) + 6y_p(t) = t^2 + 1$ if $6a$=1, $10a+6b = 0$ and $2a+5b+6c = 1$. Thus $a = \frac{1}{6}$, $b = -\frac{5}{3}a = -\frac{5}{18}$ and $c = \frac{1-2a-5b}{6} = \frac{1-\frac{1}{3}+\frac{25}{18}}{6} = \frac{\frac{18-6+25}{18}}{6} = \frac{37}{108}$. The general solution of $y''(t) + 5y'(t) + 6y(t) = t^2 + 1$ is $y(t) = \frac{1}{6}t^2 - \frac{5}{18}t + \frac{37}{108} + c_1e^{-2t} + c_2e^{-3t}$.

(variation of parameter) Suppose $y_1(t)$ and $y_2(t)$ are independent solutions of $y''(t) + p(t)y'(t) + q(t)y(t) = 0$. Then a particular solution is given by
$y_p(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(t)}dt + y_2(t)\int \frac{y_2(t)g(t)}{W(t)}dt$
where $W(t) = W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$ is the Wronskian of y_1 and y_2.

(f) $t^2y''(t) - ty'(t) - 3y(t) = 4t^2$.
First, we solve $t^2y''(t) - ty'(t) - 3y(t) = 0$. Suppose $y(t) = t^r$, we have $y'(t) = rt^{r-1}$ and $y''(t) = r(r-1)t^{r-2}$. Thus $t^2y''(t)-ty'(t)-3y(t) = (r(r-1)-r-3)t^r = (r^2 - 2r - 3)t^r$. Thus $y = t^r$ is a solution of $t^2y''(t) - ty'(t) - 3y(t) = 0$ if $r^2 - 2r - 3 = (r - 3)(r + 1) = 0$. The roots of $r^2 - 2r - 3 = 0$ are -1 and 3. Therefore the general solution is $c_1t^{-1} + c_2t^3$.
Let $y_1 = t^{-1}$ and $y_2 = t^3$ to be the solutions of $t^2y''(t) - ty'(t) - 3y(t) = 0$. $W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = t^{-1}\cdot(3t^2) - t^3\cdot(-1t^{-2}) = 4t$.
Now $g(t) = 4t^2$. We have
$\int \frac{y_2g(t)}{W(y_1,y_2)(t)}dt = \int \frac{t^3 4t^2}{4t}dt = \int t^4dt = \frac{1}{5}t^5 + c$ and $\int \frac{y_1g(t)}{W(y_1,y_2)(t)}dt = \int \frac{t^{-1}4t^2}{4t}dt = \int 1dt = t + c$. So $y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(t)}dt + y_2(t)\int \frac{y_2(t)g(t)}{W(t)}dt = -t^{-1}(\frac{1}{5}t^5 + d) + t^3(t + c) = \frac{4}{5}t^4 + ct^3 - dt^{-1}$.

(g) $y''(t) + 4y = \sec(2t)$
We will use the variation of parameter formula. We have $y_1(t) = \sin(2t)$, $y_2(t) = \cos(2t)$,
$W(y_1, y_2)(t) = y_1(t)y_2'(t)-y_2(t)y_1'(t) = \sin(2t)\cdot(-2\sin(2t))-\cos(2t)\cdot(2\cos(2t)) = -2$,
$\int \frac{y_2g(t)}{W(y_1,y_2)(t)}dt = \int \frac{\cos(2t)\sec(2t)}{-2}dt = \int \frac{\cos(2t)}{-2\cos(2t)}dt = \int \frac{-1}{2}dt = \frac{-t}{2} + c$ and
$\int \frac{y_1g(t)}{W(y_1,y_2)(t)}dt = \int \frac{\sin(2t)\sec(2t)}{-2}dt = \int \frac{\sin(2t)}{-2\cos(2t)}dt = \frac{\ln|\cos(2t)|}{4} + d$. We have used substitution $u = \cos(2t)$ and $du = -2\sin(2t)dt$.
Thus $y(t) = -\sin(2t)\cdot(\frac{-t}{2} + c) + \cos(2t)(\frac{\ln|\cos(2t)|}{4} + d) = -c\sin(2t) + d\cos(2t) + \frac{t\sin(2t)}{2} + \frac{\cos(2t)\ln|\cos(2t)|}{4}$.

(h) $y''(t) + 4y = \tan(2t)$
We will use the variation of parameter formula again. From previous example, we have $y_1(t) = \sin(2t)$, $y_2(t) = \cos(2t)$ and $W(y_1, y_2)(t) = -2$.
$\int \frac{y_2g(t)}{W(y_1,y_2)(t)}dt = \int \frac{\cos(2t)\tan(2t)}{-2}dt = \int \frac{\cos(2t)\sin(2t)}{-2\cos(2t)}dt = \int \frac{-\sin(2t)}{2}dt = \frac{\cos(2t)}{4} + c$.

$\int \frac{y_1 g(t)}{W(y_1,y_2)(t)}dt = \int \frac{\sin(2t)\tan(2t)}{-2}dt = \int \frac{\sin(2t)sin(2t)}{-2\cos(2t)}dt == \int \frac{1-\cos^2(2t)}{-2\cos(2t)}dt = \int(\frac{\cos(2t)}{2} - \frac{\sec(2t)}{2})dt = \frac{\sin(2t)}{4} - \frac{\ln|\sec(2t)+\tan(2t)|}{4} + d.$

Thus $y(t) = -\sin(2t)\cdot(\frac{\cos(2t)}{4} + c) - \cos(2t)(\frac{\sin(2t)}{4} + \frac{\ln|\sec(2t)+\tan(2t)|}{4} + d) = -\cos(2t)\frac{\ln|\sec(2t)+\tan(2t)|}{4} - c\sin(2t) + d\cos(2t).$

(i) $t^2y''(t) - t(t+2)y'(t) + (t+2)y(t) = t^4e^t(1+t).$

Given that $y_1(t) = te^t$ is a solution of $t^2y''(t) - t(t+2)y'(t) + (t+2)y(t) = 0.$

Rewrite $t^2y''(t) - t(t+2)y'(t) + 2ty(t) = t^4e^t(1+t)$ as

$y''(t) - \frac{(t+2)}{t}y'(t) + \frac{2t}{t^2}y(t) = t^2e^t(1+t).$

First, we find the solution of $y''(t) - \frac{(t+2)}{t}y'(t) + \frac{2t}{t^2}y(t) = 0.$

Let $p(t) = -\frac{t+2}{t}$. Let y be a solution of $t^2y''(t) - t(t+2)y'(t) + (t+2)y(t) = 0$. We have $(\frac{y}{y_1})' = \frac{y_1y' - y_1'y}{y_1^2} = \frac{W(t)}{y_1^2} = \frac{Ce^{-\int p(t)dt}}{(te^t)^2} = \frac{Ce^{\int \frac{t+2}{t}dt}}{t^2e^{2t}} = \frac{Ce^{\int(1+\frac{2}{t})dt}}{t^2e^{2t}} = \frac{Ce^{(t+2\ln(t))}}{t^2e^{2t}} = \frac{Ce^te^{2\ln t}}{t^2e^{2t}} = \frac{Ce^tt^2}{t^2e^{2t}} = Ce^{-t}$. So $\frac{y}{y_1} = \int Ce^{-t}dt = -Ce^{-t} + D$ and $y = y_1(-Ce^{-t} + D) = te^t(-Ce^{-t} + D) = -Ct + Dte^t$. So the general solution of $t^2y''(t) - t(t+2)y'(t) + (t+2)y(t) = 0$ is $y = -Ct + Dte^t$. We may choose the second independent solution to be $y_2 = t$.

Now $y_1 = te^t$, $y_2 = t$,$y_1' = e^t + te^t$,$y_2' = 1$ and $W(y_1,y_2)(t) = y_1y_2' - y_2y_1' = te^t\cdot 1 - t(e^t + te^t) = -t^2e^t$. Recall that $g(t) = t^2e^t(1+t)$.

$\int \frac{y_2g(t)}{W(y_1,y_2)(t)}dt = \int \frac{t\cdot t^2e^t(1+t)}{-t^2e^t}dt = \int(-t - t^2)dt = -\frac{t^2}{2} - \frac{t^3}{3} + c.$

$\int \frac{y_1g(t)}{W(y_1,y_2)(t)}dt = \int \frac{te^t\cdot t^2e^t(1+t)}{-t^2e^t}dt = \int(-te^t - t^2e^t)dt = te^t - e^t - t^2e^t + d.$

Thus $y(t) = -te^t\cdot(-\frac{t^2}{2} - \frac{t^3}{3} + c) + t(te^t - e^t - t^2e^t + d) = (\frac{1}{3}t^4 - \frac{1}{2}t^3 + t^2 - t)e^t - cte^t + dt.$

(j) $(1-t)y''(t) + ty'(t) - y(t) = 2(t-1)^2e^{-t}.$

Given that $y_1(t) = t$ is a solution of $(1-t)y''(t) + ty'(t) - y(t) = 0.$

Rewrite $(1-t)y''(t) + ty'(t) - y(t) = 2(t-1)^2e^{-t}$ as

$y''(t) + \frac{t}{1-t}y'(t) - \frac{1}{1-t}y(t) = \frac{2(t-1)^2e^{-t}}{1-t} = 2(1-t)e^{-t}.$

First, we find the solution of $y''(t) + \frac{t}{1-t}y'(t) - \frac{1}{1-t}y(t) = 0.$

Let $p(t) = \frac{t}{1-t}$. Let y be a solution of $y''(t) + \frac{t}{1-t}y'(t) - \frac{1}{1-t}y(t) = 0$. We have $(\frac{y}{y_1})' = \frac{yy_2' - y'y_2}{y_1^2} = \frac{W(t)}{y_1^2} = \frac{Ce^{-\int p(t)dt}}{t^2} = \frac{Ce^{\int \frac{-t}{1-t}dt}}{t^2} = \frac{Ce^{\int \frac{-t+1-1}{1-t}dt}}{t^2} = \frac{Ce^{\int(1+\frac{1}{t-1})dt}}{t^2e^{2t}} = \frac{Ce^{(t+\ln(t-1))}}{t^2} = \frac{Ce^{-t}(t-1)}{t^2} = \frac{Ce^t(t-1)}{t^2} = Ce^t(\frac{1}{t} - \frac{1}{t^2})$. So $\frac{y}{y_1} = \int Ce^t(\frac{1}{t} - \frac{1}{t^2})dt = C\frac{e^t}{t} + D$ and $y = y_1(C\frac{e^t}{t} + D) = t(C\frac{e^t}{t} + D) = Ce^t + Dt$. So the general solution of $y''(t) + \frac{t}{1-t}y'(t) - \frac{1}{1-t}y(t) = 0$ is $y = Ce^t + Dt$. We may choose the second independent solution to be $y_2 = e^t$.

Now $y_1 = t$, $y_2 = e^t$,$y_1' = 1$,$y_2' = e^t$ and $W(y_1,y_2)(t) = y_1y_2' - y_2y_1' = te^t - e^t = e^t(t-1)$. Recall that $g(t) = 2(1-t)e^{-t}$.

$\int \frac{y_2g(t)}{W(y_1,y_2)(t)}dt = \int \frac{e^t\cdot 2(1-t)e^{-t}}{e^t(t-1)}dt = \int(-2e^{-t})dt = 2e^{-t} + c.$

$\int \frac{y_1 g(t)}{W(y_1,y_2)(t)} dt = \int \frac{t \cdot 2(1-t)e^{-t}}{e^t(t-1)} dt = \int (-2te^{-2t}) dt = \frac{1}{2}\, e^{-2\,t}\,(2\,t+1) + d.$
Thus $y(t) = -t \cdot (2e^{-t} + c) + e^t(\frac{1}{2}\, e^{-2\,t}\,(2\,t+1) + d) = -te^{-t} + \frac{1}{2}e^{-t} - ct + de^t.$