MATH 3860 – REVIEW FOR FINAL EXAM

The final exam will be comprehensive. It will cover materials from the following sections: 1.1-1.3; 2.1-2.2;2.4-2.6;3.1-3.7; 4.1-4.3;6.1-6.6; 7.1; 7.4-7.6; 7.8.

The following is a summary of the topics which are on the review sheet. The materials after the second midterm will consists 50% $\sim$ 60% of the final exam.

The best way to study for the final exam is to do the review problems and make sure that you understand those concepts related to each problem(topic).

You should do the review problems for final exam. It would also be a good idea to go over the midterm exams and homework problems.

I. First order ODEs.

1. Linear equation: Sec 2.1, Problem 2b-2h.

To solve the linear equation $y' + p(t)y = g(t)$.: multiple the equation (on both sides) by the integrating factor

$$\mu = e^{\int p(t)dt}$$

so that the equation changes to

$$(ye^{\int p(t)dt})' = g(t)e^{\int p(t)dt} \tag{0.1}$$

Integrating, we obtain

$$y(t) = e^{-\int p(t)dt}\Big(\int g(t)e^{\int p(t)dt}dt + C\Big) \tag{0.2}$$

where C is an arbitrary constant. In the presence of an initial value condition

$$y(t_0) = y_0 \tag{0.3}$$

the value of C is determined by submitting $t = t_0$ and $y = y_0$ into formula (0.2). Suppose the functions $p(t)$ and $g(t)$ are continuous in an interval $\alpha < t < \beta$ that contains t_0 (that is, $\alpha < t_0 < \beta$). Then the solution is valid in (at least) this whole interval $\alpha < t < \beta$.

2. Separable equation: Sec2.2, Problem 2a

An equation is **separable** if it can be written in the form

$$y'(x) = g(x)h(y).$$

The solution is given (implicitly) by

$$\int \frac{dy}{h(y)} = \int g(x)dx + C.$$

Note that the roots of $h(y) = 0$, are also (constant) solutions of the equation. These constant solutions are called **equilibrium** solutions. The zeros of $h(y)$ are also called **critical points**.

3. Exact equation: Sec2.6, Problem 2n.

The equation $M(x, y)dx + N(x, y)dy = 0$bis exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

When this is the case, the solution is given by

$$F(x, y) = C.$$

Here F satisfies

$$M = \frac{\partial F}{\partial x}, \quad N = \frac{\partial F}{\partial y}.$$

To find F:

$$F(x, y) = \int M(x, y)dx + \varphi(y),$$

and

$$\varphi'(y) = N - \frac{\partial}{\partial y} \int M(x, y)dx.$$

Finally, integrate this to find φ.

4. Bernoulli equation: Hw 27 in Sec 2.4 p77, problem 2i

$y' + p(t)y = g(t)y^n$, $n \neq 0, 1$.

This can be changed to a linear equation: Let $v = y^{1-n}$. Then

$$\frac{dv}{dt} = (1 - n)y^{-n}\frac{dy}{dt}.$$

Therefore,

$$v' + (1 - n)p(t)v = (1 - n)g(t).$$

This becomes a linear equation which can be solved easily. After solving v, we get $y = v^{\frac{1}{1-n}}$.

5. Special substitution to simplify the problem: Problem 1, 2j-2m.

Use special substitution to transform a problem to a first order linear problem or seperable equation.

6. Autonomous equation: $y' = h(y)$ Sec2.5, problem 3 and 4.

Equations of the form

$$y' = h(y)$$

are called **autonomous equations**. This is a special case of separable equations. Here we are interested in behaviors of solutions, especially when t is large or goes to infinity, in terms of the initial value $y(0) = y_0$. Note that the roots of $h(y) = 0$, are also (constant) solutions of the equation. These constant solutions are called **equilibrium** solutions.

This can be done without solving the equation. Important concepts here are *equilibrium solutions*, *stability/instability* (asymptotically stable, unstable, semistable solutions). To do the stability analysis, one has to
(a) Find the zeroes of $h(y) = 0$.
(b) Find the interval where $h(y)$ is positive or negative.
Now we can understand the behavior of the solution from the fact that y is increasing if $h(y) > 0$ and y is decreasing if $h(y) < 0$.

We are also expected to find the limit of a solution according to its initial value.

7. Initial value problem. The solution to a first order equation is not unique – it contains an arbitrary constant C. To find a specific solution satisfying a given initial value condition

$$y(t_0) = y_0,$$

one substitute this into the solution to determine the value of the constant C.

II. Second order Linear ODEs. The standard form:

$$y'' + p(t)y' + q(t)y = g(t). \tag{0.4}$$

Note that here the coefficient of the y'' term is 1. A lot of formulas in the book are for this standard form.

1. Structure of the solution space. The general solution of (0.4) has the form

$$y = y_p + c_1 y_1 + c_2 y_2$$

where y_p is a particular solution of (0.4) and $c_1y_1+c_2y_2$ represents the general solution of the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{0.5}$$

This is based on the following linearity properties of the equations:

- The sum of a solution of (0.4) and a solution of (0.5) is a solution of (0.4).
- The difference of two solutions of (0.4) is a solution of (0.5).
- Any linear combination of two solutions of (0.5) is again a solution of (0.5).

Consequently, there are two steps in finding the general solution of (0.4):

(1) Find the general solution of the corresponding homogeneous equation (0.5).
(2) Find a particular solution of (0.4). (This means any solution of (0.4).)

2. The homogeneous equation: $g(t) \equiv 0$. The general solution of the homogeneous equation (0.5) is of the form

$$y = c_1y_1 + c_2y_2.$$

Here y_1, y_2 are any two *linearly independent* solutions; we call such two solutions a *fundamental set* of solutions. (So any two linearly independent solutions form a fundamental set.)

Linear independence and the Wronskian.

- The *Wronskian* of y_1 and y_2 is defined as

  $$W(y_1, y_2) = y_1y_2' - y_1'y_2 = \det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

- **Abel's Theorem.** The Wronskian $W = W(y_1, y_2)$ of two solutions y_1, y_2 of (0.5) satisfies the first order linear equation

  $$W' + p(t)W = 0.$$

  Consequently,

  $$W(y_1, y_2) = Ce^{-\int p(t)dt},$$

  which is either identically zero (if $C = 0$) or never zero (if $C \neq 0$).
- Two solutions y_1, y_2 of (0.5) are linearly independent (i.e, they form a fundamental set of solutions) if and only if $W(y_1, y_2)$ is nonzero.

Homogeneous equation of constant coefficients. Sec 3.1-3.5. Problem 5a-5c
The general solution of

$$y'' + by' + cy = 0$$

is dependent of the roots of the *characteristic equation*:

$$r^2 + br + c = 0. \tag{0.6}$$

Let r_1, r_2 denote the roots of (0.6). Then

$$y(t) = \begin{cases} c_1e^{r_1t} + c_2e^{r_2t}, & r_1 \neq r_2, \text{ both real} \\ c_1e^{r_1t} + c_2te^{r_1t}, & r_1 = r_2, \text{ real} \\ c_1e^{\alpha t}\cos\beta t + c_2e^{\alpha t}\sin\beta t, & r_1 = \alpha + i\beta,\ \beta \neq 0. \end{cases}$$

It is interesting to recall how we derive these formulas.

Euler equation. Sec5.5 Problem 3f-3i

Euler equation is the differential equation of the form

$$t^2y''(t) + \alpha ty'(t) + \beta y(t) = 0.$$

Suppose $y(t) = t^r$, we have $y'(t) = rt^{r-1}$ and $y''(t) = r(r-1)t^{r-2}$. Thus $t^2y''(t) + \alpha ty'(t) + \beta y(t) = (r(r-1) + \alpha r + \beta)t^r = (r^2 + r(\alpha-1) + \beta)t^r$. So $y = t^r$ is a solution of $t^2y''(t) + \alpha ty'(t) + \beta y(t) = 0$ if

$$r^2 + r(\alpha - 1) + \beta = 0. \tag{0.7}$$

Let r_1, r_2 denote the roots of (0.7). Then

$$y(t) = \begin{cases} c_1t^{r_1} + c_2t^{r_2}, & r_1 \neq r_2, \text{ both real} \\ c_1t^{r_1} + c_2t^{r_1}\ln(t), & r_1 = r_2, \text{ real} \\ c_1t^{\alpha}\cos(\beta\ln t) + c_2t^{\alpha}\sin(\beta\ln t), & r_1 = \alpha + i\beta, r_2 = \alpha - i\beta,\ \beta \neq 0. \end{cases}$$

Note that $t^{\alpha+i\beta} = t^{\alpha}t^{i\beta} = t^{\alpha}e^{i\beta\ln t} = t^{\alpha}(\cos(\beta\ln t) + i\sin(\beta\ln t))$
$= t^{\alpha}\cos(\beta\ln t) + it^{\alpha}\sin(\beta\ln t)$.

Methods to find the general solution when one solution is known. Problem 6a.

For the general homogeneous equation (0.5), if one solution y_1 is given, then we can find a second linear independent solution (with y_1) by using the **Abel Theorem** as follows: Let y_2 be an arbitrary solution of (0.5). By Abel's Theorem

$$(\frac{y_2}{y_1})' = \frac{y_2'y_1 - y_2y_1'}{y_1^2} = \frac{Ce^{-\int p(t)dt}}{y_1^2}$$

Integrating this we obtain

$$y_2 = c_1 y_1 + C y_1 \int (e^{-\int p(t)dt}/y_1^2)dt,$$

which gives the **general solution** of (0.5).

3. Methods to find particular solutions. Sec 3.6 and 3.7, Problem 6, 7, 8, 9

Method of Undetermined Coefficients. Here we deal with nonhomogeneous equations with constant coefficients: $p(x) = b$ and $q(x) = c$ are constant in equation (0.4). So we know how to find the general solution of the corresponding homogeneous equation

$$y'' + by' + cy = 0. \tag{0.8}$$

To find a particular solution of the *nonhomogeneous* equation $y'' + by' + cy = g(t)$, we first decide what such a solution must look like by using the method of annihilator. The following is important in finding annihilator. We have $(D-a)^k(t^{k-1}e^{at}) = 0$, $((D-a)^2 + b^2)^k(t^{k-1}e^{at}\cos(bt)) = 0$ and $((D-a)^2 + b^2)^k(t^{k-1}e^{at}\sin(bt)) = 0$.

Then we determine undetermined coefficients by plugging it to the equation.

Variation of parameters. This is a method to find the general solution of the *nonhomogeneous* equation (0.4) when we know two linearly independent solutions of the corresponding homogeneous equation (0.5). It works as follows.

Let $y_1(t)$ and $y_2(t)$ be linearly independent solutions of the *homogeneous* equation (0.5). We want to find solutions of the *nonhomogeneous* equation (0.4) of the form

$$y = -y_1 \int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}dt + y_2 \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}dt. \tag{0.9}$$

Here $W(y_1, y_2)(t) = y_1 y_2' - y_1' y_2$ is the Wrosnkian.

4. Initial value problems. The general solution of (0.4) or (0.5) contains two arbitrary constants c_1 and c_2. As in the case of first order equation, to find a specific solution satisfying the given initial value conditions

$$y(t_0) = y_0, \;\; y'(t_0) = y_0'$$

one substitute this into the general solution to determine the values of c_1 and c_2 by solving the resulting 2×2 linear system for c_1 and c_2.

III. Higher Order Linear Equation. Sec4.1-Sec4.3, Problem 5d, 5e.
The solution of a $n-th$ order linear equation

$$a_0y^n(t)+a_1y^{n-1}(t)+\cdots+a_{n-1}y'(t)+a_ny(t) = (a_0D^n+a_1D^{n-1}+\cdots+a_{n-1}D+a_n)y(t) = 0$$

is determined by the roots of the characteristic equation $a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n = 0$.
If $r = a$ is a root of multiplicity k of the characteristic equation then $\{e^{at}, te^{at}, \cdots, t^{k-1}e^{at}\}$ are solutions.
If $r = a \pm ib$ is a root of multiplicity k of the characteristic equation then $\{e^{at}\cos(bt), e^{at}\sin(bt), te^{at}\cos(bt), te^{at}\sin(bt), \cdots, t^{k-1}e^{at}\cos(bt), t^{k-1}e^{at}\sin(bt)\}$ are solutions. Note that a $n-th$ order linear equation has n independent solutions.

Method of annihilator: Problem 8.
We have $(D-a)^k(t^{k-1}e^{at}) = 0$,
$((D-a)^2+b^2)^k(t^{k-1}e^{at}\cos(bt)) = 0$ and $((D-a)^2+b^2)^k(t^{k-1}e^{at}\sin(bt)) = 0$.

IV. Laplace Transform. Ch 6, Problem 5j-5n, Problem 10.
One can take the Laplace transform of the differential equation to get the Laplace transform of the solution of a differential equation. Then one use inverse Laplace transform to find the solution. One important technique is to express solution in terms of convolution integral.

V. First order system. Problem 11, 12, 13, 15, 16.
(a) Solving first order linear system, stability analysis of the linear system and the sketch of trajectories of the system.

(b) Stability analysis of the critical point of the nonlinear system and use the slope fields to analyze the behavior of the solution. Problem 4, 5.

(c) Variation of parameters: Understand the meaning of the formula.