Solutions to HW 10

(1) (Sec 6.2 Problem 21)(30 pts) $y'' - 2y' + 2y = \cos(t)$. $y(0) = 1$, $y'(0) = 0$

Taking the Laplace's transform of the differential equation $y'' - 2y' + 2y = \cos(t)$,we have
$L(y'' - 2y' + 2y) = L(\cos(t)) = \frac{s}{s^2+1}$. Using $L(y'') = s^2L(y) - sy(0) - y'(0)$ and $L(y') = sL(y) - y(0)$, we have
$s^2L(y) - sy(0) - y'(0) - 2(sL(y) - y(0)) + 2L(y) = \frac{s}{s^2+1}$ and
$(s^2 - 2s + 2)L(y) - sy(0) - y'(0) + 2y(0) = \frac{s}{s^2+1}$.
Substituting $y(0) = 1$ and $y'(0) = 0$, we have
$(s^2 - 2s + 2)L(y) - s + 2 = \frac{s}{s^2+1}$ and
$(s^2 - 2s + 2)L(y) = s - 2 + \frac{s}{s^2+1}$.
So $L(y) = \frac{s-2}{s^2-2s+2} + \frac{s}{(s^2-2s+2)(s^2+1)}$.

Note that $s^2 - 2s + 2 = (s - 1)^2 + 1$. First, we simplify the term by noting that $\frac{s-2}{s^2-2s+2} = \frac{s-2}{(s-1)^2+1} = \frac{-1}{(s-1)^2+1} + \frac{3}{(s-1)^2+1}$.

Now we simplify the term $\frac{s}{(s^2-2s+2)(s^2+1)}$ by partial fraction. We have $\frac{s}{(s^2-2s+2)(s^2+1)} = \frac{s}{((s-1)^2+1)(s^2+1)} = \frac{a(s-1)+b}{(s-1)^2+1} + \frac{cs+d}{s^2+1}$. Multiplying $((s - 1)^2 + 1)(s^2 + 1)$, we have
$s = (a(s - 1) + b)(s^2 + 1) + (cs + d)((s - 1)^2 + 1)$
$= (as + (-a + b))(s^2 + 1) + (cs + d)(s^2 - 2s + 2)$
$= as^3 + (-a + b)s^2 + as + (-a + b) + cs^3 + ds^2 - 2cs^2 - 2ds + 2cs + 2d$
$= (a + c)s^3 + ((-a + b) + d - 2c)s^2 + (a - 2d + 2c)s + (-a + b + 2d)$. Comparing the coefficient, we get $a+c = 0$, $-a+b+d-2c = 0$, $a-2d+2c = 1$ and $-a+b+2d = 0$. From $a + c = 0$, we have $c = -a$, $a + b + d = 0$, $-a - 2d = 1$ and $-a + b + 2d = 0$. Multiplying -1 to $a + b + d = 0$, we get $-a - b - d = 0$. Adding $-a + b + 2d = 0$, we have $-2a + d = 0$. Using $-a - 2d = 1$ and $-2a + d = 0$, we have $a = -\frac{1}{5}$ and $d = -\frac{2}{5}$. From $a + b + d = 0$ and $c = -a$, we get $b = -a - d = \frac{3}{5}$ and $c = \frac{1}{5}$. Hence $\frac{s}{(s^2-2s+1)(s^2+1)} = -\frac{1}{5}\frac{(s-1)}{(s-1)^2+1} + \frac{3}{5}\frac{1}{(s-2)^2+1} + \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1}$.

From $L(y) = \frac{s-2}{s^2-2s+2} + \frac{s}{(s^2-2s+2)(s^2+1)}$,
$\frac{s-2}{s^2-2s+2} = \frac{s-1}{(s-1)^2+1} + \frac{-1}{(s-1)^2+1}$ and
$\frac{s}{(s^2-2s+1)(s^2+1)} = -\frac{1}{5}\frac{(s-1)}{(s-1)^2+1} + \frac{3}{5}\frac{1}{(s-2)^2+1} + \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1}$, we get
$L(y) = \frac{4}{5}\frac{(s-1)}{(s-1)^2+1} - \frac{2}{5}\frac{1}{(s-1)^2+1} + \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1}$ and
$y(t) = \frac{4}{5}e^t\cos(t) - \frac{2}{5}e^t\sin(t) + \frac{1}{5}\cos(t) - \frac{2}{5}\sin(t)$.

(2) (Sec 6.2 Problem 23)(30 pts) $y'' + 2y' + y = 4e^{-t}$ $y(0) = 2$, $y'(0) = -1$

Taking the Laplace's transform of the differential equation $y'' + 2y' + y = 4e^{-t}$, we have $L(y'' + 2y' + y) = L(4e^{-t})$. Using $L(y'') = s^2L(y) - sy(0) - y'(0)$, $L(y') = sL(y) - y(0)$ and $L(4e^{-t}) = 4L(e^{-t}) = \frac{4}{s+1}$, we have
$s^2L(y) - sy(0) - y'(0) + 2(sL(y) - y(0)) + L(y) = \frac{4}{s+1}$ and

$(s^2+2s+1)L(y) - sy(0) - y'(0) - 2y(0) = \frac{4}{s+1}$.
Substituting $y(0) = 2$ and $y'(0) = -1$, we have
$(s^2+2s+1)L(y) - s\cdot 2 - (-1) - 2\cdot 2 = \frac{4}{s+1}$ and $(s^2+4s+4)L(y) = \frac{4}{s+1} + 2s + 3$.
Note that $s^2+2s+1 = (s+1)^2$. So we have $(s+1)^2 L(y) = \frac{4}{s+1} + 2s + 3$ and $L(y) = \frac{4}{(s+1)^3} + \frac{2s+3}{(s+1)^2}$. We can simplify $\frac{2s+3}{(s+1)^2}$ by substituting $u = s+1$, $s = u-1$ and $\frac{2s+3}{(s+1)^2} = \frac{2(u-1)+3}{u^2} = \frac{2u+1}{u^2} = \frac{2}{u} + \frac{1}{u^2} = \frac{2}{(s+1)} + \frac{1}{(s+1)^2}$.

Hence we have $L(y) = \frac{4}{(s+1)^3} + \frac{2}{(s+1)} + \frac{1}{(s+1)^2}$ and
$y = L^{-1}(\frac{4}{(s+1)^3} + \frac{2}{(s+1)} + \frac{1}{(s+1)^2})$.
Using $L(e^{-t}) = \frac{1}{s+1}$, $L(te^{-t}) = \frac{1}{(s+1)^2}$, and $L(t^2e^{-t}) = \frac{2}{(s+1)^3}$,
we have $L^{-1}(\frac{1}{s+1}) = e^{-t}$, $L^{-1}(\frac{1}{(s+1)^2}) = te^{-t}$ and $L^{-1}(\frac{1}{(s+1)^3}) = \frac{t^2e^{-t}}{2}$.
Therefore
$y = L^{-1}(\frac{4}{(s+1)^3} + \frac{2}{(s+1)} + \frac{1}{(s+1)^2})$
$= 4\cdot\frac{t^2e^{-t}}{2} + 2e^{-t} + te^{-t} = 2t^2e^{-t} + 2e^{-t} + te^{-t}$.

(3) (Sec 6.3 Problem 8)(15 pts)

$$f(t) = \begin{cases} 0, & 0 \le t < 1, \\ t^2 - 2t + 2, & 1 \le t. \end{cases}$$

We have $f(t) = (t^2 - 2t + 2)u_1(t)$. Let $g(t-1) = t^2 - 2t + 2 = (t-1)^2 + 1$. Then $g(t) = g((t+1)-1) = t^2 - 1$ and $f(t) = u_1(t)g(t-1)$. Hence $L(f(t)) = L(u_1(t)g(t-1)) = e^{-s}L(g(t)) = e^{-s}L(t^2+1) = e^{-s}(\frac{2}{s^3} + \frac{1}{s})$.

(4) (Sec 6.3 Problem 9)(15 pts)

$$f(t) = \begin{cases} 0, & 0 \le t < \pi, \\ t - \pi, & \pi \le t < 2\pi, \\ 0, & 2\pi \le t. \end{cases}$$

We have $f(t) = (t-\pi)u_{\pi,2\pi}(t) = (t-\pi)(u_\pi(t) - u_{2\pi}(t)) = (t-\pi)u_\pi(t) - (t-\pi)(u_{2\pi}(t)$. Let $g(t-\pi) = t - \pi$ and $h(t-2\pi) = t - \pi$ Then $g(t) = g((t+\pi)-\pi) = t$, $h(t) = h((t+2\pi)-2\pi) = t + 2\pi - \pi = t + \pi$ and $f(t) = u_\pi(t)g(t-\pi) - u_{2\pi}(t)g(t-2\pi)$. Hence $L(f(t)) = L(u_\pi(t)g(t-\pi) - u_{2\pi}(t)h(t-2\pi)) = e^{-\pi s}L(g(t)) - e^{-2\pi s}L(h(t)) = e^{-\pi s}L(t) - e^{-2\pi s}L(t+\pi) = e^{-\pi s}\frac{1}{s^2} - e^{-2\pi s}(\frac{1}{s^2} + \pi\frac{1}{s})$.

(5) (Sec 6.3 Problem 10)(10 pts) $f(t) = u_1(t) + 2u_2(t) - 6u_4(t)$. $L(f(t)) = L(u_1(t) + 2u_2(t) - 6u_4(t)) = L(u_1(t)) + 2L(u_2(t)) - 6L(u_4(t)) = \frac{e^{-s}}{s} + 2\frac{e^{-2s}}{s} - 6\frac{e^{-4s}}{s}$.