Solutions to HW 11

(1) (Sec 6.4 Problem 1) (15 pts) $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 1$

$$f(t) = \begin{cases} 1, & 0 \leq t < \frac{\pi}{2}, \\ 0, & \frac{\pi}{2} \leq t. \end{cases}$$

We have $f(t) = u_0(t) - u_1(t)$ and $L(f(t)) = \frac{1}{s} - \frac{e^{-s}}{s}$.

Since $L(y''+y) = L(f(t)) = \frac{1}{s} - \frac{e^{-s}}{s}$ and $L(y''+y) = (s^2+1)Y(s) - sy(0) - y'(0) = (s^2+1)Y(s) - 1$, we have $(s^2+1)Y(s) - 1 = \frac{1}{s} - \frac{e^{-s}}{s}$, $(s^2+1)Y(s) = \frac{1}{s} - \frac{e^{-s}}{s} + 1$ and $Y(s) = \frac{1}{s(s^2+1)} - \frac{e^{-s}}{s(s^2+1)} + \frac{1}{(s^2+1)}$. Using partial fraction, we have $\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{(s^2+1)}$ and $g(t) = L^{-1}(\frac{1}{s(s^2+1)}) = 1 - \cos(t)$.

So $y(t) = 1 - \cos(t) + u_{\frac{\pi}{2}}(t)f(t - \frac{\pi}{2}) + \sin(t) = 1 - \cos(t) + u_{\frac{\pi}{2}}(t)(1 - \cos(t - \frac{\pi}{2}) + \sin(t) = 1 - \cos(t) + u_{\frac{\pi}{2}}(t)(1 - \sin(t)) + \sin(t)$.

Note that we have used the fact that $\cos(t - \frac{\pi}{2}) = \sin(t)$.

(2) (Sec 6.4 Problem 5) (15 pts) $y'' + 3y' + 2y = f(t)$, $y(0) = 0$, $y'(0) = 0$

$$f(t) = \begin{cases} 1, & 0 \leq t < 10, \\ 0, & 10 \leq t. \end{cases}$$

We have $f(t) = u_0(t) - u_{10}(t)$ and $L(f(t)) = \frac{1}{s} - \frac{e^{-10s}}{s}$.

Since $L(y'' + 3y' + 2y) = L(f(t)) = \frac{1}{s} - \frac{e^{-10s}}{s}$ and $L(y'' + 3y' + 2y) = (s^2 + 3s + 2)Y(s) - sy(0) - y'(0) + 3y(0) = (s^2 + 3s + 2)Y(s)Y(s)$, we have $(s^2 + 3s + 2)Y(s) = \frac{1}{s} - \frac{e^{-10s}}{s}$, and $Y(s) = \frac{1}{s(s^2+3s+2)} - \frac{e^{-10s}}{s(s^2+3s+2)}$. Using partial fraction, we have $\frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} = \frac{a}{s} + \frac{b}{(s+1)} + \frac{c}{(s+2)}$ Multiplying $s(s+1)(s+2)$, we get $1 = a(s+1)(s+2) + bs(s+2) + cs(s+1)$. Plugging in $s = 0$, we have $1 = 2a$ and $a = \frac{1}{2}$. Plugging in $s = -1$, we have $1 = -b$ and $b = -1$. Plugging in $s = -2$, we have $1 = -2c$ and $c = -\frac{1}{2}$. So $\frac{1}{s(s^2+3s+2)} = \frac{1}{2}\frac{1}{s} - \frac{1}{(s+1)} - \frac{1}{2}\frac{1}{(s+2)}$ and $g(t) = L^{-1}(\frac{1}{s(s^2+3s+2)}) = L^{-1}(\frac{1}{2}\frac{1}{s} - \frac{1}{(s+1)} - \frac{1}{2}\frac{1}{(s+2)}) = \frac{1}{2} - e^{-t} - \frac{1}{2}e^{-2t}$.

Hence $y(t) = L^{-1}(\frac{1}{s(s^2+3s+2)} - \frac{e^{-10s}}{s(s^2+3s+2)}) = g(t) - u_{10}(t)g(t-10)$
$= \frac{1}{2} - e^{-t} - \frac{1}{2}e^{-2t} - u_{10}(t)(\frac{1}{2} - e^{-(t-10)} - \frac{1}{2}e^{-2(t-10)})$.

(3) (Sec 6.4 Problem 9)(20 pts) $y'' + y = g(t)$, $y(0) = 0$, $y'(0) = 1$

$$g(t) = \begin{cases} \frac{t}{2}, & 0 \leq t < 6, \\ 3, & 6 \leq t. \end{cases}$$

We have $g(t) = \frac{t}{2}(u_0(t) - u_6(t)) + 3u_6(t) = \frac{t}{2} + (2 - \frac{t}{2})u_6(t)$ and $L(g(t)) = L(\frac{t}{2}) + L((2 - \frac{t}{2})u_6(t))$. Let $f(t-6) = 2 - \frac{t}{2}$. We have $f(t) = -\frac{t}{2}$.
$L((2 - \frac{t}{2})u_6(t)) = L(f(t-6)u_6(t)) = e^{6s}L(f(t)) = -\frac{e^{6s}}{2s^2}$
and $L(g(t)) = \frac{1}{2s^2} - \frac{e^{6s}}{2s^2}$.

Taking the Laplace's transform of the equation, we have $L(y''+y) = L(g(t)) = \frac{1}{2s^2} - \frac{e^{6s}}{2s^2}$. Since $L(y''+y) = (s^2+1)Y(s) - sy(0) - y'(0) = (s^2+1)Y(s) - 1$, we have $(s^2+1)Y(s) - 1 = \frac{1}{2s^2} - \frac{e^{6s}}{2s^2}$, $Y(s) = \frac{1}{s^2+1} + \frac{1}{2s^2(s^2+1)} - \frac{e^{6s}}{2s^2(s^2+1)}$ and $y(t) = L^{-1}(\frac{1}{s^2+1} + \frac{1}{2s^2(s^2+1)} - \frac{e^{6s}}{2s^2(s^2+1)})$. Using partial fraction, we have $\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$ and $f(t) = L^{-1}(\frac{1}{s^2(s^2+1)}) = L^{-1}(\frac{1}{s^2}) - L^{-1}(\frac{1}{s^2+1}) = t - \sin(t)$.

So $y(t) = L^{-1}(\frac{1}{s^2+1}) + L^{-1}(\frac{1}{2s^2(s^2+1)}) - L^{-1}(\frac{e^{6s}}{2s^2(s^2+1)})$
$= \sin(t) + \frac{1}{2}(t - \sin(t)) - \frac{1}{2}f(t-6)u_6(t) = \frac{1}{2}(t + \sin(t)) - \frac{1}{2}(t - 6 - \sin(t-6))u_6(t)$.

(4) (Sec 6.5 Problem 2) (15 pts)$y''(t) + 4y(t) = \delta(t-\pi) - \delta(t-2\pi)$, with $y(0) = 0$ and $y'(0) = 0$.

Taking the Laplace transform $L(y''(t) + 4y(t)) = L(\delta(t-\pi) - \delta(t-2\pi))$, we have $(s^2+4)Y(s) = e^{-\pi s} - e^{-2\pi s}$.
$\Rightarrow Y(s) = \frac{e^{-\pi s}}{(s^2+4)} + \frac{e^{-2\pi s}}{(s^2+4)}$.

Let $f(t) = L^{-1}(\frac{1}{s^2+4}) = \frac{\sin(2t)}{2}$.

We have $y(t) = L^{-1}(\frac{e^{-\pi s}}{(s^2+4)}) + L^{-1}(\frac{e^{-2\pi s}}{(s^2+4)}) = u_\pi(t)f(t-\pi) + u_{2\pi}(t)f(t-2\pi) = u_\pi(t)\frac{\sin(2(t-\pi))}{2} + u_{2\pi}(t)\frac{\sin(2(t-\pi))}{2} = \frac{u_\pi(t)\sin(2t)}{2} - \frac{u_{2\pi}(t)\sin(2t)}{2}$

(5) (Sec 6.5 Problem 3) (15 pts)$y''(t) + 3y'(t) + 2y(t) = \delta(t-5) + u_{10}(t)$, with $y(0) = 0$ and $y'(0) = \frac{1}{2}$.

Taking the Laplace transform $L(y''(t) + 3y'(t) + 2y(t)) = L(\delta(t-5)) + L(u_{10}(t))$, we have $(s^2+3s+2)Y(s) - y'(0) = e^{-5s} + \frac{e^{-10s}}{s}$ and
$Y(s) = \frac{1}{2(s^2+3s+2)} + \frac{e^{-5s}}{(s^2+3s+2)} + \frac{e^{-10s}}{s(s^2+3s+2)}$.

We have $\frac{1}{2(s^2+3s+2)} = \frac{a}{s+1} + \frac{b}{s+2}$, $\frac{1}{2} = a(s+2) + b(s+1)$. Plugging $s = -1$, we have $a = \frac{1}{2}$. Plugging $s = -2$, we have $b = -\frac{1}{2}$. So $\frac{1}{2(s^2+3s+2)} = \frac{1}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s+2}$ and $L^{-1}(\frac{1}{2(s^2+3s+2)}) = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t}$.

By partial fraction, we have $\frac{1}{(s^2+3s+2)} = \frac{a}{s+1} + \frac{b}{s+2}$, $1 = a(s+2) + b(s+1)$. Plugging $s = -1$, we have $a = 1$. Plugging $s = -2$, we have $b = -1$. So $\frac{1}{(s^2+3s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$ and $f(t) = L^{(}\frac{1}{(s^2+3s+2)}) = e^{-t} - e^{-2t}$.

By partial fraction, we have $\frac{1}{s(s^2+3s+2)} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{s+2}$, $1 = a(s+1)(s+2) + bs(s+2) + cs(s+1)$. Plugging $s = 0$, we have $a = \frac{1}{2}$. Plugging $s = -1$, we have $b = -1$. Plugging $s = -2$, we have $c = \frac{1}{2}$.So $\frac{1}{s(s^2+3s+2)} = \frac{1}{2}\frac{1}{s} - \frac{1}{s+1} - \frac{1}{2}\frac{1}{s+2}$ and $g(t) = L^{(}\frac{1}{s(s^2+3s+2)}) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$.

Now $y(t) = L^{-1}(\frac{1}{2(s^2+3s+2)} + \frac{e^{-5s}}{(s^2+3s+2)} + \frac{e^{-10s}}{s(s^2+3s+2)})$
$= \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t} + u_5(t)f(t-5) + u_{10}(t)g(t-10)$
$= \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t} + u_5(t)(= e^{-(t-5)} - e^{-2(t-5)}) + u_{10}(t)(\frac{1}{2} - e^{-(t-10)} + \frac{1}{2}e^{-2(t-10)})$.

(6) (Sec 6.5 Problem 11) (20 pts)$y''(t) + 2y'(t) + 2y(t) = \cos(t) + \delta(t - \frac{\pi}{2})$, with $y(0) = 0$ and $y'(0) = 0$.

Taking the Laplace transform $L(y''(t) + 2y'(t) + 2y(t)) = L(\cos(t)) + L(\delta(t - \frac{\pi}{2}))$, we have $(s^2 + 2s + 2)Y(s) = \frac{s}{s^2+1} + e^{-\frac{\pi}{2}s}$.

$\Rightarrow Y(s) = \frac{s}{(s^2+2s+2)(s^2+1)} + \frac{e^{-\frac{\pi}{2}s}}{(s^2+2s+2)}$

Let $f(t) = L^{-1}(\frac{1}{s^2+2s+2}) = L^{-1}(\frac{1}{(s+1)^2+1}) = e^{-t}\sin(t)$.

By partial fraction, we have $\frac{s}{(s^2+2s+2)(s^2+1)} = \frac{as+b}{s^2+1} + \frac{c(s+1)+d}{(s+1)^2+1}$.
Multiplying $(s^2+2s+2)(s^2+1)$, we have $s = (as+b)(s^2+2s+2)+(cs+c+d)(s^2+1) = (a+c)s^3 + (2a+b+c+d)s^2 + (2b+2a+c)s + 2b+d+c$. Comparing the coefficients, we have $a+c = 0$, $2a+b+c+d = 0$, $2b+2a+c = 1$ and $2b+d+c = 0$. Using $a = -c$, we have $-c+b+d = 0$, $2b-c = 1$ and $2b+d+c = 0$. From $2b-c = 1$, we have $c = 2b-1$, $-2b+1+b+d = 0$ and $2b+d+2b-1 = 0$. Solving $-b+d+1 = 0$ and $4b+d-1 = 0$, we have $d = -\frac{3}{5}$ and $b = \frac{2}{5}$, $c = 2b-1 = -\frac{1}{5}$, $a = -c = \frac{1}{5}$.
Hence $\frac{s}{(s^2+2s+2)(s^2+1)} = \frac{\frac{1}{5}s+\frac{2}{5}}{s^2+1} + \frac{-\frac{1}{5}(s+1)-\frac{3}{5}}{(s+1)^2+1}$.

Recall that $Y(s) = \frac{s}{(s^2+2s+2)(s^2+1)} + \frac{e^{-\frac{\pi}{2}s}}{(s^2+2s+2)}$.
We have $y(t) = L^{-1}(\frac{\frac{1}{5}s+\frac{2}{5}}{s^2+1} + \frac{-\frac{1}{5}(s+1)-\frac{3}{5}}{(s+1)^2+1}) + L^{-1}(\frac{e^{-\frac{\pi}{2}s}}{(s^2+2s+2)}) = \frac{1}{5}\cos(t) + \frac{2}{5}\sin(t) - \frac{1}{5}e^{-t}\cos(t) - \frac{3}{5}e^{-t}\sin(t) + u_{\frac{\pi}{2}}(t)f(t - \frac{\pi}{2})$
$= \frac{1}{5}\cos(t) + \frac{2}{5}\sin(t) - \frac{1}{5}e^{-t}\cos(t) - \frac{3}{5}e^{-t}\sin(t) + u_{\frac{\pi}{2}}(t)e^{-(t-\frac{\pi}{2})}\sin(t - \frac{\pi}{2})$.