MATH 3860 Solution to HW 2

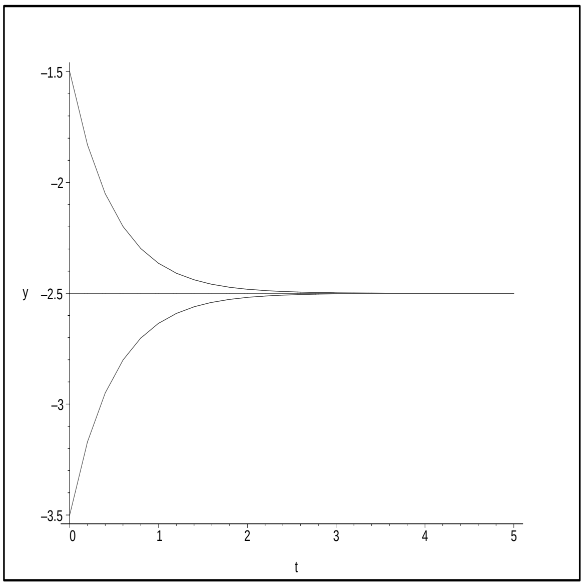

FIGURE 1.

1. (5 pts) (Problem 5 from sec1.3) $\frac{d^2y}{dt^2} + \sin(t+y) = \sin(t)$ is a nonlinear O.D.E.

2. (5 pts) (Problem 6 from sec1.3) $\frac{d^3y}{dt^3} + t\frac{dy}{dt} + \cos^2(t)y = t^3$ is a linear O.D.E.

3. (5 pts) (Problem 9 from sec1.3) $ty'(t) - y = t^2$ is a linear O.D.E.

4. (5 pts) (Problem 12 from sec1.3)$t^2y''(t) + 5ty' + 4y = 0$ is a linear O.D.E.

5. (15 pts) (Sec1.2 (p15) 1b) From $\frac{dy}{dt} = -2y + 5$, we have $\int \frac{dy}{-2y+5} = \int dt$. So $\frac{\ln|-2y+5|}{-2} = t + c$, $-2y + 5 = Ce^{-2t}$ and $y = \frac{-5}{2} + Ce^{-2t}$. Using $y(0) = y_0$, we have $y_0 = \frac{-5}{2} + C$ and $C = y_0 + \frac{5}{2}$. So $y(t) = \frac{-5}{2} + (y_0 + \frac{5}{2})e^{-2t}$. We have $\lim_{t\to\infty} y(t) = \frac{-5}{2}$. See figure 1 for the graph of the solution.

6. (15 pts)(Sec 2.2 Problem 21). From $\frac{dy}{dx} = \frac{(1+3x^2)}{3y^2-6y}$, we have $\int 3y^2 - 6ydy = \int 1 + 3x^2dx$. So $y^3 - 3y^2 = x + x^3 + c$. Using $y(0) = 1$, we get $1 - 3 = 0 + c$ and $c = -2$. The solution satisfies $y^3 - 3y^2 = x + x^3 - 2$.

From $\frac{dy}{dx} = \frac{(1+3x^2)}{3y^2-6y}$, we know the solution doesn't exist when $3y^2 - 6y = 3y(y-2) = 0$, i.e. $y = 0$ or $y = 2$. Using $y^3 - 3y^2 = x + x^3 - 2$ and $y = 0$, we have $x + x^3 - 2 = 0$. Factoring $x + x^3 - 2 = 0$, we get $(x-1)(x^2+x+2) = 0$. Thus $x = 1$ is the only real root of $x + x^3 - 2 = 0$.

Using $y^3 - 3y^2 = x + x^3 - 2$ and $y = 2$, we have $x + x^3 + 2 = 0$. Factoring $x + x^3 + 2 = 0$, we get $(x+1)(x^2 - x + 2) = 0$. Thus $x = -1$ is the only real root of $x + x^3 + 2 = 0$. So the solution exists if $-1 < x < 1$.

7. (20 pts)(Sec 2.2 Problem 30)

(a) $\frac{dy}{dx} = \frac{y-4x}{x-y} = \frac{\frac{y-4x}{x}}{\frac{x-y}{x}} = \frac{\frac{y}{x}-4}{1-\frac{y}{x}}$

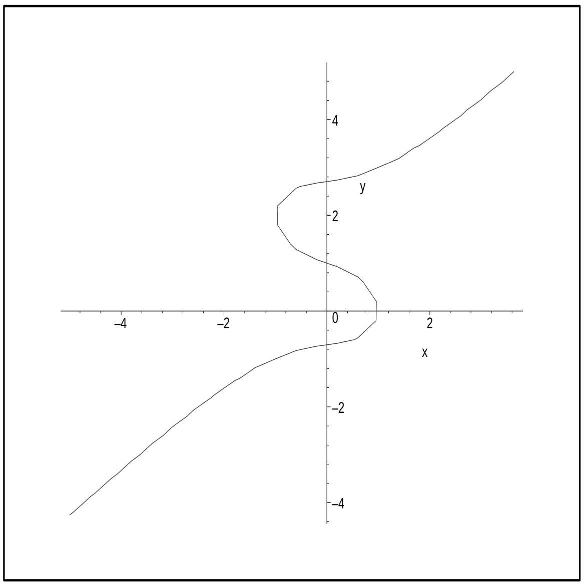

FIGURE 2. $y^3 - 3y^2 = x + x^3 - 2$

(b) Let $v = \frac{y}{x}$. Then $y = vx$ and $\frac{dy}{dx} = \frac{d}{dx}(vx) = \frac{dv}{dx}x + v$.

(c) Using $v = \frac{y}{x}$ and $\frac{dy}{dx} = \frac{dv}{dx}x + v$, we can rewrite the equation $\frac{dy}{dx} = \frac{\frac{y}{x}-4}{1-\frac{y}{x}}$ as $\frac{dv}{dx}x + v = \frac{v-4}{1-v}$.

We can simplify

$\frac{dv}{dx}x + v = \frac{v-4}{1-v}$

as $\frac{dv}{dx}x = \frac{v-4}{1-v} - v = \frac{v-4-v+v^2}{1-v} = \frac{v^2-4}{1-v}$.

So $\int \frac{1-v}{v^2-4}dv = \int \frac{dx}{x}$.

(d)

Let $\frac{1-v}{v^2-4} = \frac{a}{v-2} + \frac{b}{v+2}$.

Multiplying $(v-2)(v+2)$, we have $1 - v = a(v+2) + b(v-2)$. Plugging $v = 2$, we have $-1 = 4a$ and $a = -\frac{1}{4}$. Plugging $v = -2$, we have $3 = -4b$ and $b = -\frac{3}{4}$.

Thus we have $\frac{1-v}{v^2-4} = -\frac{1}{4}\frac{1}{v-2} + -\frac{3}{4}\frac{1}{v+2}$ and $\int \frac{1-v}{v^2-4}dv = \int(-\frac{1}{4}\frac{1}{v-2} + -\frac{3}{4}\frac{1}{v+2})dv = -\frac{1}{4}\ln|v-2| - \frac{3}{4}\ln|v+2| + C$.

From $\int \frac{1-v}{v^2-4}dv = \int \frac{dx}{x}$, we have

$-\frac{1}{4}\ln|v-2| - \frac{3}{4}\ln|v+2| = \ln|x| + C$.

(e) Note that

$-\frac{1}{4}\ln|v-2| - \frac{3}{4}\ln|v+2| = \ln|v-2|^{-\frac{1}{4}} + \ln|v+2|^{-\frac{3}{4}} = \ln|v-2|^{-\frac{1}{4}}|v+2|^{-\frac{3}{4}}$.

So the eq $-\frac{1}{4}\ln|v-2| - \frac{3}{4}\ln|v+2| = \ln|x| + C$ can be simplified as $\ln|v-2|^{-\frac{1}{4}}|v+2|^{-\frac{3}{4}} = \ln|x| + c$ and $|v-2|^{-\frac{1}{4}}|v+2|^{-\frac{3}{4}} = Cx$.

Use $v = \frac{y}{x}$. Then $|v-2|^{-\frac{1}{4}}|v+2|^{-\frac{3}{4}} = |\frac{y}{x}-2|^{-\frac{1}{4}}|\frac{y}{x}+2|^{-\frac{3}{4}} = |\frac{y-2x}{x}|^{-\frac{1}{4}}|\frac{y+2x}{x}|^{-\frac{3}{4}}$ $= |(y-2x)^{-\frac{1}{4}}(y+2x)^{-\frac{3}{4}}x|$.

So $|v-2|^{-\frac{1}{4}}|v+2|^{-\frac{3}{4}} = C|x|$ can be simplified as $|(y-2x)^{-\frac{1}{4}}(y+2x)^{-\frac{3}{4}}| = C$.

8. (15 pts)(Sec 2.1 Problem 10) $ty' - y = t^2e^{-t}$.

Dividing the equation by t, we get $y' - \frac{1}{t}y = te^{-t}$.

We have $p(t) = -\frac{1}{t}$ and $g(t) = te^{-t}$.

The integrating factor is $\mu(t) = e^{\int p(t)dt} = e^{\int -\frac{1}{t}dt} = e^{-\ln t} = e^{\ln t^{-1}} = t^{-1}$. So $y(t) = \frac{\int \mu(t)g(t)dt}{\mu(t)} = \frac{\int t^{-1}te^{-t}dt}{t^{-1}} = \frac{\int e^{-t}dt}{t^{-1}} = \frac{-e^{-t}+C}{t^{-1}} = -te^{-t} + Ct$. Note that $\lim_{t\to\infty} te^{-t} = 0$. So $\lim_{t\to\infty} y(t) = \infty$ if $C > 0$, $\lim_{t\to\infty} y(t) = -\infty$ if $C < 0$ and $\lim_{t\to\infty} y(t) = 0$ if $C = 0$.

9. (15 pts)(Sec 2.1 Problem 31) $y' - \frac{3}{2}y = 3t + 2e^t$.

We have $p(t) = -\frac{3}{2}$ and $g(t) = 3t + 2e^t$.

The integrating factor is $\mu(t) = e^{\int p(t)dt} = e^{\int -\frac{3}{2}dt} = e^{-\frac{3}{2}t}$. So $y(t) = \frac{\int \mu(t)g(t)dt}{\mu(t)} = \frac{\int e^{-\frac{3}{2}t}\cdot(3t+2e^t)dt}{e^{-\frac{3}{2}t}} = \frac{\int(3te^{-\frac{3}{2}t}+2e^{-\frac{1}{2}t})dt}{e^{-\frac{3}{2}t}}$.

Note that $\int 3te^{-\frac{3}{2}t}dt = -\frac{4}{3}e^{-\frac{3}{2}t} - 2te^{-\frac{3}{2}t}$ and $\int 2e^{-\frac{1}{2}t}dt = -4e^{-\frac{1}{2}t} + C$.

So $\int(3te^{-\frac{3}{2}t} + 2e^{-\frac{1}{2}t})dt = -\frac{4}{3}e^{-\frac{3}{2}t} - 2te^{-\frac{3}{2}t} - 4e^{-\frac{1}{2}t} + C$

and $y(t) = \frac{\int(3te^{-\frac{3}{2}t}+2e^{-\frac{1}{2}t})dt}{e^{-\frac{3}{2}t}} = \frac{-\frac{4}{3}e^{-\frac{3}{2}t}-2te^{-\frac{3}{2}t}-4e^{-\frac{1}{2}t}+C}{e^{-\frac{3}{2}t}} = -\frac{4}{3} - 2t - 4e^t + Ce^{\frac{3}{2}t}$.

Using $y(0) = y_0$, we get $y_0 = -\frac{4}{3} - 4 + C$ and $C = y_0 + \frac{16}{3}$.

Thus $y(t) = -\frac{4}{3} - 2t - 4e^t + (y_0 + \frac{16}{3})e^{\frac{3}{2}t}$.

If $y_0 + \frac{16}{3} > 0$ then $\lim_{t\to\infty} y(t) = \infty$.

If $y_0 + \frac{16}{3} < 0$ then $\lim_{t\to\infty} y(t) = -\infty$.