MATH 3860 Solution to HW 4

1. (20 pts) (Problem 9 from sec 2.5) Let $f(y) = y^2(y^2 - 1)$. So $f(y) = 0$ if $y = 0$ or $y = \pm 1$. We have $f(-2) = (-2)^2((-2)^2 - 1) > 0$,$f(-0.5) = (-0.5)^2((-0.5)^2 - 1) < 0$, $f(0.5) = (0.5)^2((0.5)^2 - 1) < 0$ and $f(2) = (2)^2((2)^2 - 1) > 0$

The equilibrium points are $y = 0$, $y = 1$ and $y = -1$.

We also have $f(y) > 0$ when $y \in (-\infty, -1) \cup (1, \infty)$ and $f(y) < 0$ when $y \in (-1, 0) \cup (0, 1))$. Therefore -1 is a asymptotically stable equilibrium points, 1 is an unstable equilibrium point and 0 is a semistable equilibrium point.

2. (20 pts) (Problem 10 from sec 2.5) Let $f(y) = y(1 - y^2)$. So $f(y) = 0$ if $y = 0$ or $y = \pm 1$. We have $f(-2) = (-2)(1-(-2)^2) > 0$,$f(-0.5) = (-0.5)(1-(-0.5)^2) < 0$, $f(0.5) = (0.5)(1 - (0.5)^2) > 0$ and $f(2) = (2)(1 - (2)^2) < 0$

The equilibrium points are $y = 0$, $y = 1$ and $y = -1$.

We also have $f(y) > 0$ when $y \in (-\infty, -1) \cup (0, 1)$ and $f(y) < 0$ when $y \in (-1, 0) \cup (1, \infty))$. Therefore -1 and 1 are asymptotically stable equilibrium points and 0 is an unstable equilibrium point.

3. (20 pts) (Problem 21 from sec 2.5)

(a) Let $f(y) = r(1 - \frac{y}{K})y - h = -\frac{r}{K}y^2 + ry - h$. So $f(y) = 0$ if $y = \frac{-r \pm \sqrt{r^2 - 4\frac{rh}{K}}}{-2\frac{r}{K}} = \frac{K}{2} \pm \frac{K\sqrt{1-4\frac{h}{rK}}}{2}$. If $1 - 4\frac{h}{rK} > 0$ then the given eq has two equilibrium points $y_1 = \frac{K}{2} - \frac{K\sqrt{1-4\frac{h}{rK}}}{2}$ and $y_2 = \frac{K}{2} + \frac{K\sqrt{1-4\frac{h}{rK}}}{2}$.

(b) We know that $f(y) < 0$ if $y \in (\infty, y_1) \cup (y_2, \infty)$ and $f(y) > 0$ if $y \in (y_1, y_2)$. So y_1 is unstable and y_2 is asymptotically stable.

(c)From the sign graph of $f(y)$, we know that if $y(0) = y_0 < y_1$ then $y(t)$ is decreasing to $-\infty$ and $y(t)$ decreases to 0 in finite time, if $y_1 < y(0) = y_0 < y_2$ then $y(t)$ is increasing to y_2 and $y(0) = y_0 > y_2$ then $y(t)$ is decreasing to y_2.

(d) If $h > 1 - \frac{rK}{4}$ then $f(y)$ has no real root, then $f(y) < 0$ for all y. So $y'(t) = f(y) < 0$ and $y(t)$ is decreasing to 0 as t increases.

(e) If $h = 1 - \frac{rK}{4}$ then $f(y)$ has repeated roots $y = \frac{K}{2}$. The equilibrium point is semistable.

We also have $f(y) > 0$ when $y \in (-\infty, -1) \cup (0, 1)$ and $f(y) < 0$ when $y \in (-1, 0) \cup (1, \infty))$. Therefore -1 and 1 are asymptotically stable equilibrium points and 0 is an unstable equilibrium point.

4. (20 pts) (Problem 3 from sec 2.6) $3x^2 - 2xy + 2 + (6y^2 - x^2 + 3)\frac{dy}{dx} = 0$

Let $M = 3x^2 - 2xy + 2$ and $N = 6y^2 - x^2 + 3$. Note that $M_y = -2x$ and $N_x = -2x$. Now $M_y = N_x$. This equation is exact. Solving $F_x = 3x^2 - 2xy + 2$, we have $F = \int(3x^2 - 2xy + 2)dx = x^3 - x^2y + g(y)$. Using $F_y = N = 6y^2 - x^2 + 3$, we get
$(x^3 - x^2y + g(y))_y = 6y^2 - x^2 + 3$ and $-x^2 + g'(y) = 6y^2 - x^2 + 3$. So $g'(y) = 6y^2 + 3$ and $g(y) = \int 6y^2 + 3dy = 2y^3 + 3y + c$. Hence $F(x, y) = x^3 - x^2y + 2y^3 + 3y$ and the solution satisfies $F(x, y) = x^3 - x^2y + 2y^3 + 3y = C$.

5. (20 pts) (Problem 10 from sec 2.6) $\frac{y}{x} + 6x + (\ln(x) - 2)\frac{dy}{dx} = 0$

Let $M = \frac{y}{x} + 6x$ and $N = \ln(x) - 2$. Note that $M_y = \frac{1}{x}$ and $N_x = \frac{1}{x}$. Now $M_y = N_x$. This equation is exact. Solving $F_x = \frac{y}{x} + 6x$, we have $F = \int(\frac{y}{x} + 6x)dx = y\ln(x) + 3x^2 + g(y)$. Using $F_y = N = \ln(x) - 2$, we get $(y\ln(x) + 3x^2 + g(y))_y = \ln(x) - 2$ and $\ln(x) + g'(y) = \ln(x) - 2$. So $g'(y) = -2$

and $g(y) = 2y + c$. Hence $F(x, y) = y\ln(x) + 3x^2 + 2y$ and the solution satisfies $F(x, y) = y\ln(x) + 3x^2 + 2y = C$.