MATH 3860 Solution to HW 5

1. (15 pts) (Problem 4 from sec 3.1) $2y''(t) - 3y'(t) + y = 0$. The characteristic equation of $2y''(t) - 3y'(t) + y = 0$ is $2r^2 - 3r + 1 = (2r-1)(r-1) = 0$. We have $r = \frac{1}{2}$ or $r = 1$. Thus the general solution is $y(t) = c_1 e^{\frac{1}{2}t} + c_2 e^t$.

2. (15 pts) (Problem 10 from sec 3.1) The characteristic equation of $y''(t) + 4y'(t) + 3y(t) = 0$ is $r^2 + 4r + 3 = (r+1)(r+3) = 0$. We have $r = -1$ or $r = -3$. Thus the general solution is $y(t) = c_1 e^{-t} + c_2 e^{-3t}$.

Computing $y'(t)$, we get $y'(t) = -c_1 e^{-t} - 3c_2 e^{-3t}$

Using $y(0) = 2$ and $y'(0) = -1$, we have

$c_1 e^0 + c_2 e^0 = 2$ and $-c_1 e^0 - 3c_2 e^0 = -1$, i.e $c_1 + c_2 = 2$ and $-c_1 - 3c_2 = -1$. We have $c_1 = \frac{5}{2}$ and $c_2 = -\frac{1}{2}$. So $y(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$.

3. (15 pts) (Problem 21 from sec 3.1) The characteristic equation of $y''(t) - y'(t) - 2y(t) = 0$ is $r^2 - r - 2 = (r+1)(r-2) = 0$. We have $r = -1$ or $r = 2$. Thus the general solution is $y(t) = c_1 e^{-t} + c_2 e^{2t}$.

Computing $y'(t)$, we get $y'(t) = -c_1 e^{-t} + 2c_2 e^{2t}$

Using $y(0) = \alpha$ and $y'(0) = 2$, we have

$c_1 e^0 + c_2 e^0 = \alpha$ and $-c_1 e^0 + 2c_2 e^0 = 2$, i.e $c_1 + c_2 = \alpha$ and $-c_1 + 2c_2 = 2$. We have $c_1 = \frac{2\alpha-2}{3}$ and $c_2 = \frac{\alpha+2}{3}$. So $y(t) = \frac{2\alpha-2}{3}e^{-t} + \frac{\alpha+2}{3}e^{2t}$. Since $\lim_{t\to\infty} e^{-t} = 0$ and $\lim_{t\to\infty} e^{2t} = \infty$, the solution $y(t) = \frac{2\alpha-2}{3}e^{-t} + \frac{\alpha+2}{3}e^{2t}$ approaches 0 only if $\frac{\alpha+2}{3} = 0$, i.e $\alpha = -2$.

4. (15 pts) (Problem 24 from sec 3.1) The characteristic equation of $y''(t) + (3-\alpha)y'(t) - 2(\alpha-1)y(t) = 0$ is $r^2 + (3-\alpha)r - 2(\alpha-1) = (r - \alpha + 1)(r+2) = 0$. We have $r = \alpha - 1$ or $r = -2$. Thus the general solution is $y(t) = c_1 e^{(\alpha-1)t} + c_2 e^{-2t}$. If $\alpha < 1$, then $\lim_{t\to\infty} y(t) = 0$.

Since $\lim_{t\to\infty} e^{-2t} = 0$, there is no α such that $y(t)$ is unbounded.

5. (15 pts) (Problem 9 from Sec 3.2) Rewrite the equation $t(t-4)y''(t) + 3ty'(t) + 4y(t) = 2$ as $y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$. So $p(t) = \frac{3}{t-4}$, $q(t) = \frac{4}{t(t-4)}$ and $g(t) = \frac{2}{t(t-4)}$. Hence $p(t)$ is continuous if $t \in (-\infty, 4) \cup (4, \infty)$, $q(t)$ is continuous if $t \in (-\infty, 0) \cup (0, 4) \cup (4, \infty)$ and $g(t)$ is continuous if $t \in (-\infty, 0) \cup (0, 4) \cup (4, \infty)$. Therefore $p(t)$, $q(t)$ and $g(t)$ are continuous if $t \in (-\infty, 0) \cup (0, 4) \cup (4, \infty)$. The initial conditions are $y(3) = 0$ and $y'(3) = -1$. We have $3 \in (0, 4)$. Thus the solution exists on the interval $(0, 4)$.

6. (10 pts) (Problem 16 from Sec 3.2)

Since $y(t) = \sin(t^2)$, we have $y(0) = 0$, $y'(t) = 2t\cos(t^2)$ and $y'(0) = 0$, By the uniqueness of the solution of homogeneous equation, we must have $y(t) = 0$. This means that $y(t) = \sin(t^2)$ can't be a solution of $y'' + p(t)y' + q(t)y = 0$.

7. (15 pts) (Sec 3.2 Problem 25) Solution: Since $y_1(x) = x$ and $y_2(x) = xe^x$, we have $y_1'(x) = 1$, $y_1''(x) = 0$, $y_2'(x) = e^x + xe^x$and $y_2''(x) = e^x + e^x + xe^x = 2e^x + xe^x$. So $x^2y_1'' - x(x+2)y_1' + (x+2)y_1 = 0 - x(x+2) + (x+2)x = 0$ and $x^2y_2'' - x(x+2)y_2' + (x+2)y_2 = x^2(2e^x + xe^x) - x(x+2)(e^x + xe^x) + (x+2)xe^x =$

$2x^2e^x + x^3e^x - x^2e^x - x^3e^x - 2xe^x - 2x^2e^x + x^2e^x + 2xe^x = 0$. So y_1 and y_2 are solutions of $x^2y'' - x(x+2)y' + (x+2)y = 0$. The Wronskain of y_1 and y_2 is $W(y_1, y_2)(x) = y_1y_2' - y_2y_1' = x \cdot (e^x + xe^x) - xe^x \cdot 1 = x^2e^x > 0$ if $x > 0$. Hence y_1 and y_2 form a set of fundamental solutions.