MATH 3860 Solution to HW 6

1. (10 pts) (Sec 3.3 Problem 15)
Solution: Rewrite the equation as $y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 0$. Let $p(t) = -\frac{t+2}{t} = -1 - \frac{2}{t}$. Now the Wronskian is $W(t) = ce^{-\int p(t)dt} = ce^{-\int(-1-\frac{2}{t})dt} = ce^{\int(1+\frac{2}{t})dt} = ce^{t+2\ln t} = ce^t e^{2\ln t} = ce^t t^2$.

2. (10 pts) (Sec 3.3 Problem 18)
Solution: Rewrite the equation as $y'' - \frac{2x}{1-x^2}y' + \frac{\alpha(\alpha+1)}{1-x^2}y = 0$. Let $p(x) = -\frac{2x}{1-x^2}$. Now the Wronskian is $W(x) = ce^{-\int p(x)dx} = ce^{-\int(-\frac{2x}{1-x^2})dx} = ce^{\int(\frac{2x}{1-x^2})dx} = ce^{-\ln(1-x^2)} = \frac{c}{1-x^2}$. We have used the substitution $u = 1 - x^2$ and $du = -2xdx$ to get $\int(\frac{2x}{1-x^2})dx = -\int \frac{du}{u} = -\ln(u) + c = -\ln(1-x^2) + c$.

3. (15 pts) (Sec 3.3 Problem 20)
Solution: Rewrite the equation as $y'' + \frac{2}{t}y' + e^t y = 0$. Let $p(t) = \frac{2}{t}$. Now the Wronskian is $W(y_1, y_2)(t) = ce^{-\int p(t)dt} = ce^{-\int \frac{2}{t}dt} = ce^{\int(\frac{2}{t})dt} = ce^{2\ln t} = ce^{2\ln t} = ct^2$. Using $W(y_1, y_2)(1) = 2$, we have $W(y_1, y_2)(1) = c \cdot 1^2 = c$. So $c = 1$, $WW(y_1, y_2)(t) = t^2$ and $W(y_1, y_2)(5) = 5^2 = 25$.

4. (10 pts)(Sec 3.3 Problem 24)
Solution: Suppose y_1 and y_2 have zero at $t = t_0$, then the Wronskian $W(t_0) = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) = 0$. By Abel Theorem, we know that $W(t) = 0$ Thus y_1 and y_2 can't be a set of fundamental solutions.

5. (10 pts)(Sec 3.4 Problem 11)
Solution:Solving $r^2 + 6r + 13 = 0$, we have $r = -3 \pm 2i$. Note that $e^{(-3+2i)t} = e^{-3t}e^{i2t} = e^{-3t}\cos(2t) + ie^{-3t}\sin(2t)$
Thus the general solution is $y(t) = c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t)$.

6. (15 pts)(Sec 3.4 Problem 18)
Solution:Solving $r^2 + 4r + 5 = 0$, we have $r = -2 \pm i$. Note that $e^{(-2+i)t} = e^{-2t}e^{it} = e^{-2t}\cos(t) + ie^{-2t}\sin(t)$.
Thus the general solution is $y(t) = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t)$.
Taking the derivative of y, we get
$y'(t) = -2c_1 e^{-2t}\cos(t) - c_1 e^{-2t}\sin(t) - 2c_2 e^{-2t}\sin(t) + c_2 e^{-2t}\cos(t)$
$= (-2c_1 + c_2)e^{-2t}\cos(t) + (-c_1 - 2c_2)e^{-2t}\sin(t)$. Using $y(0) = 1$ and $y'(0) = 0$, we have $c_1 = 1$ and $-2c_1 + c_2 = 0$. So $c_2 = 2c_1 = 2$. Hence $y(t) = e^{-2t}\cos(t) + 2e^{-2t}\sin(t)$.

7. (15 pts)(Sec 3.5 Problem 12)
Solution:Solving $r^2 - 6r + 9 = (r-3)^2 = 0$, we have repeated roots $r = 3$.
Thus the general solution is $y(t) = c_1 e^{3t} + c_2 te^{3t}$.
Taking the derivative of y, we get
$y'(t) = 3c_1 e^{3t} + c_2 e^{3t} + 3c_2 te^{3t} = (3c_1 + c_2)e^{3t} + 3c_2 te^{3t}$.

Using $y(0) = 0$ and $y'(0) = 2$, we have $c_1 = 0$ and $3c_1 + c_2 = 2$. So $c_2 = 2$. Hence $y(t) = 2te^{3t}$.

8. (15 pts)((Sec 3.5 Problem 28))
Solution: Rewrite the equation $(x-1)y'' - xy' + y = 0$ as $y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = 0$ Let y be a solution of $(x-1)y'' - xy' + y = 0$. By reduction of order, we have $(\frac{y}{y_1})' = \frac{Ce^{\int \frac{x}{x-1}dx}}{(e^x)^2} = \frac{Ce^{\int(1+\frac{1}{x-1})dx}}{e^{2x}} = \frac{Ce^{(x+\ln(x-1))}}{e^{2x}} = \frac{Ce^x(x-1)}{e^{2x}} = C(x-1)e^{-x}$. So $\frac{y}{y_1} = \int C(x-1)e^{-x} = -Cxe^{-x} + D$ and $y = y_1(-Cxe^{-x} + D) = e^x(-Cxe^{-x} + D) = -Cx + De^x$. So the second independent solution is x.