Solution to HW 7

1. (10 pts) ((Sec 5.5 Problem 1))
$x^2y''(x)+4xy'(x)+2y(x)=0$. Suppose $y(t)=x^r$, we have $y'(x)=rx^{r-1}$ and $y''(x)=r(r-1)x^{r-2}$. Thus $x^2y''(x)+4xy'(x)+2y(x)=(r(r-1)+4r+2)x^r=(r^2+3r+2)x^r$. Thus $y=x^r$ is a solution of $x^2y''(x)+4xy'(x)+2y(x)=0$ if $r^2+3r+2=(r+1)(r+2)=0$. The roots of $r^2+3r+2=0$ are -1 and -2. Therefore the general solution is $y(x)=c_1x^{-1}+c_2x^{-2}$.

2. (10 pts) ((Sec 5.5 Problem 3)) $x^2y''(x)-3xy'(x)+4y(x)=0$. Suppose $y(x)=x^r$, we have $y'(x)=rx^{r-1}$ and $y''(x)=r(r-1)x^{r-2}$. Thus $x^2y''(x)-3xy'(x)+4y(x)=(r(r-1)-3r+4)x^r=(r^2-4r+4)x^r$. Thus $y=x^r$ is a solution of $x^2y''(x)-3xy'(x)+4y(x)=0$ if $r^2-4r+4=(r-2)^2=0$. The roots of $r^2-4r+4=0$ are -2 and -2. Therefore the general solution is $y(x)=c_1x^{-2}+c_2x^{-2}\ln|x|$.

3. (10 pts) ((Sec 5.5 Problem 4))
$x^2y''(x)+3xy'(x)+5y(x)=0$. Suppose $y(x)=x^r$, we have $y'(x)=rx^{r-1}$ and $y''(x)=r(r-1)x^{r-2}$. Thus $x^2y''(x)+3xy'(x)+5y(x)=(r(r-1)+3r+5)x^r=(r^2+2r+5)x^r$. Thus $y=x^r$ is a solution of $x^2y''(x)+3xy'(x)+5y(x)=0$ if $r^2+2r+5=0$. The roots of $r^2+2r+5=0$ are $-1+2i$ and $-1-2i$. Note that $x=e^{\ln x}$ and $x^{-1+2i}=x^{-1}x^{2i}=x^{-1}e^{i2\ln x}=x^{-1}\cos(2\ln x)+ix^{-1}\sin(2\ln x)$. Therefore the general solution is $c_1x^{-1}\cos(2\ln x)+c_2x^{-1}\sin(2\ln x)$.

4. (10 pts)((Sec 3.6 Problem 2))
Solution:Solving $r^2+2r+5=0$,we have $r=-1\pm 2i$. The solution of $y''(t)+2y'(t)+5y=0$ is $y(t)=c_1e^{-t}\sin(2t)+c_2e^{-t}\cos(2t)$. We try $y_p=c\sin(2t)+d\cos(2t)$ to be a particular solution of $y''(t)+2y'(t)+5y=3\sin(2t)$. We have $y_p=c\sin(2t)+d\cos(2t)$,
$y_p'=2c\cos(2t)-2d\sin(2t)$,
$y_p''=-4c\sin(2t)-4d\cos(2t)$ and $y_p''(t)+2y_p'(t)+5y_p(t)=-4c\sin(2t)-4d\cos(2t)+4c\cos(2t)-4d\sin(2t)+5c\sin(2t)+5d\cos(2t)=(c-4d)\sin(2t)+(4c+d)\cos(2t)=3\sin(2t)$ if $c-4d=3$, $4c+d=0$, $c=\frac{3}{17}$ and $d=-\frac{12}{17}$. Thus the general solution of $y''(t)+2y'(t)+5y=3\sin(2t)$ is $y(t)=\frac{3}{17}\sin(2t)+-\frac{12}{17}\cos(2t)+c_1e^{-t}\sin(2t)+c_2e^{-t}\cos(2t)$.

5. (15 pts)((Sec 3.6 Problem 14)) Solving $r^2+4=0$, we know that the solution of $y''(t)+4y=0$ is $y(t)=c_1\sin(2t)+c_2\cos(2t)$. We try $y_p=ct^2+dt+e+fe^t$ to be a particular solution of $y''(t)+4y=t^2+3e^t$. We have $y_p=ct^2+dt+e+fe^t$,
$y_p'=2ct+d+fe^t$,
$y_p''=2c+fe^t$ and $y_p''(t)+4y_p(t)=2c+fe^t+4ct^2+4dt+4e+4fe^t=4ct^2+4dt+(2c+4e)+5fe^t=t^2+3e^t$ if $4c=1$, $4d=0$, $2c+4e=0$ and $5f=3$. So $c=\frac{1}{4}$, $d=0$, $e=-\frac{1}{8}$ and $f=\frac{3}{5}$.

Thus the general solution of is $y(t) = \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t + c_1\sin(2t) + c_2\cos(2t)$. Using the initial condition $y(0) = 0$ and $y'(0) = 2$, we get $y(0) = -\frac{1}{8} + \frac{3}{5} + c_2 = 0$, $y'(0) = \frac{3}{5} + 2c_1 = 2$, $c_1 = \frac{7}{10}$ and $c_2 = -\frac{19}{40}$. Thus $y(t) = \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t + \frac{7}{10}\sin(2t) - \frac{19}{40}\cos(2t)$.

6. (15 pts)((Sec 3.6 Problem 17)) Solution: Solving $r^2 + 4 = 0$, we know that the solution of $y''(t) + 4y = 0$ is $y(t) = c_1\sin(2t) + c_2\cos(2t)$. We try $y_p = ct\sin(2t) + dt\cos(2t)$ to be a particular solution of $y''(t) + 4y = 3\sin(2t)$. We have $y_p = ct\sin(2t) + dt\cos(2t)$,
$y_p' = c\sin(2t) + 2ct\cos(2t) + d\cos(2t) - 2dt\sin(2t)$,
$y_p'' = 4c\cos(2t) - 4ct\sin(2t) - 4d\sin(2t) - 4dt\cos(2t)$ and $y_p''(t) + 4y_p(t) = 4c\cos(2t) - 4d\sin(2t) = 3\sin(2t)$ if $c = 0$ and $d = -\frac{3}{4}$. Thus the general solution of $y''(t) + 4y = 3\sin(2t)$ is $y(t) = -\frac{3}{4}t\cos(2t) + c_1\sin(2t) + c_2\cos(2t)$. Using the condition $y(0) = 2$ and $y'(0) = -1$, we get $c_1 = -\frac{1}{8}$ and $c_2 = 2$. Thus $y(t) = -\frac{3}{4}t\cos(2t) - \frac{1}{8}\sin(2t) + 2\cos(2t)$.

7. (15 pts)(Sec 3.7 Problem 5)
Solution: Solving $r^2 + 1 = 0$, we have $r = \pm i$. So the general solution of $y''(t) + y(t) = 0$ is $y(t) = c_1\cos(t) + c_2\sin(t)$. We will use the variation of parameter formula to solve $y''(t) + y(t) = \tan(t)$. We have $y_1(t) = \cos(t)$, $y_2(t) = \sin(t)$, $g(t) = \tan(t)$
$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = \cos(t)\cdot(\cos(t)) - \sin(t)\cdot(-\sin(t)) = \cos^2(t) + \sin^2(t) = 1$,
$\int \frac{y_2 g(t)}{W(y_1,y_2)(t)}dt = \int \frac{\sin(t)\tan(t)}{1}dt = \int \frac{\sin^2(t)}{\cos(t)}dt = \int \frac{1-\cos^2(t)}{\cos(t)}dt = \int \sec(t) - \cos(t)dt = \ln|\sec(t) + \tan(t)| - \sin(t) + c$ and
$\int \frac{y_1 g(t)}{W(y_1,y_2)(t)}dt = \int \frac{\cos(t)\tan(t)}{1}dt = \int \sin(t)dt = -\cos(t) + d$. We have Thus $y(t) = -\cos(t)\cdot(\ln|\sec(t) + \tan(t)| - \sin(t) + c) + \sin(t)(-\cos(t) + d)$.

8. (15 pts)(Sec 3.7 Problem 14)
Solution: Rewrite $t^2y''(t) - t(t+2)y'(t) + (t+2)y(t) = 2t^3$ as
$y''(t) - \frac{t+2}{t}y'(t) + \frac{t+2}{t^2}y(t) = 2t$.
We will use the variation of parameter formula to solve$y''(t) - \frac{t+2}{t}y'(t) + \frac{t+2}{t^2}y(t) = 2t$. We have $y_1(t) = t$, $y_2(t) = te^t$, $g(t) = 2t$
$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = t(e^t + te^t) - te^t\cdot 1 = t^2e^t$,
$\int \frac{y_2 g(t)}{W(y_1,y_2)(t)}dt = \int \frac{te^t\cdot 2t}{t^2e^t}dt = \int 2dt = 2t + c$ and
$\int \frac{y_1 g(t)}{W(y_1,y_2)(t)}dt = \int \frac{t\cdot 2t}{t^2e^t}dt = \int 2e^{-t}dt = -2e^{-t} + d$. We have Thus $y(t) = -t\cdot(2t + c) + te^t(-2e^{-t} + d) = -2t^2 - ct - 2t + dte^t = -2t^2 - (c+2)t + dte^t$. So $y_p(t) = -2t^2$ or $y_p(t) = -2t^2 - 2t$.