Solution to HW 9

1. (Sec 4.1 Problem 6) (15 pts) We rewrite the equation $(x^2-4)y^6(t)+x^2y'''+9y=0$ as $y^6(t)+\frac{x^2}{x^2-4}y'''+\frac{9}{x^2-4}y=0$. Now the function $\frac{x^2}{x^2-4}$ and $\frac{9}{x^2-4}$ are continuous on $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$. So the solution exists on $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$.

2. (Sec 4.2 Problem 12)(15 pts) $y^{(3)}(t)-3y^{(2)}(t)+3y'(t)-y=0$.

The characteristic equation of $y^{(3)}(t)-33y^{(2)}(t)+3y'(t)-y=0$ is $r^3-3r^2+3r-1=(r^3-1)-3r(r-1)=(r-1)(r^2+r+1)-3r(r-1)=(r-1)(r^2+r+1-3r)=(r-1)(r^2-2r+1)=(r-1)^3=0$. So $r=3$ is a root of characteristic equation of order 3. The general solution is $y(t)=c_1e^t+c_2te^t+c_3t^2e^t$.

3. (Sec 4.2 Problem 15)(25 pts) The characteristic equation of $y^{(6)}(t)+y=0$ is $r^6+1=0$. So $r^6=-1=e^{i(\pi+2k\pi)}$ where k is an integer. Now $r=e^{\frac{i(\pi+2k\pi)}{6}}$ for $k=0$, 1, 2, 3, 4 and 5. So $r=e^{i(\frac{\pi}{6})}=\cos(\frac{\pi}{6})+i\sin(\frac{\pi}{6})=\frac{\sqrt{3}}{2}+i\frac{1}{2}$, $r=e^{i(\frac{3\pi}{6})}=e^{i(\frac{\pi}{2})}=\cos(\frac{\pi}{2})+i\sin(\frac{\pi}{2})=i$, $r=e^{i(\frac{5\pi}{6})}==\cos(\frac{5\pi}{6})+i\sin(\frac{5\pi}{6})=-\frac{\sqrt{3}}{2}+i\frac{1}{2}$, $r=e^{i(\frac{7\pi}{6})}==\cos(\frac{7\pi}{6})+i\sin(\frac{7\pi}{6})=-\frac{\sqrt{3}}{2}-i\frac{1}{2}$, $r=e^{i(\frac{9\pi}{6})}==\cos(\frac{3\pi}{2})+i\sin(\frac{3\pi}{2})=-i$, $r=e^{i(\frac{11\pi}{6})}==\cos(\frac{11\pi}{6})+i\sin(\frac{11\pi}{6})=\frac{\sqrt{3}}{2}-i\frac{1}{2}$. So $r=\frac{\sqrt{3}}{2}\pm i\frac{1}{2}$, $\pm i$ and $r=-\frac{\sqrt{3}}{2}\pm i\frac{1}{2}$. Thus the general solution is $y(t)=c_1e^{\frac{\sqrt{3}t}{2}}\cos(\frac{t}{2})+c_2e^{\frac{\sqrt{3}t}{2}}\sin(\frac{t}{2})+c_3\cos(t)+c_4\sin(t)+c_5e^{\frac{-\sqrt{3}t}{2}}\cos(\frac{t}{2})+c_6e^{\frac{-\sqrt{3}t}{2}}\sin(\frac{t}{2})$.

4. (Sec 4.2 Problem 22) (20 pts) $y^{(4)}(t)+2y^{(2)}(t)+y=0$.

The characteristic equation of $y^{(4)}(t)+2y^{(2)}(t)+y=0$ is $r^4+2r^2+1=(r^2+1)^2$. Its roots are $r=\pm i$ with multiplicity 2. The general solution is $y(t)=c_1\cos(t)+c_2\sin(t)+c_3t\cos(t)+c_4t\sin(t)$.

5. (Sec 4.2 Problem 29)(25 pts) $y^{'''}(t)+y^{'}(t)=0$ with $y(0)=0$, $y'(0)=1$ and $y''(0)=2$

The characteristic equation of $y^{'''}(t)+y^{'}(t)=0$ is $r^3+r=r(r^2+1)$. Its roots are$r=\pm i$ and $r=0$ The general solution is $y(t)=c_1\cos(t)+c_2\sin(t)+c_3$. So $y'(t)=-c_1\sin(t)+c_2\cos(t)$ and $y''(t)=-c_1\cos(t)-c_2\sin(t)$. Using $y(0)=0$, $y'(0)=1$, $y''(0)=2$, $\cos(0)=1$ and $\sin(0)=0$, we have $c_1+c_3=0$, $c_2=1$ and $-c_1=2$. Hence $c_1=-2$, $c_2=1$ and $c_3=-c_1=2$. The solution to the IVP is $y(t)=-2\cos(t)+\sin(t)+2$