Problem Set #1

Due: Wednesday, September 9

1. Let $N = (0, \cdots, 0, 1)$ be the "north pole" and $S = -N$ the "south pole." Define stereographic projection $\sigma : S^n \backslash \{N\} \mapsto R^n$ by

$$\sigma(x_1, \cdots, x_{n+1}) = \frac{(x_1, \cdots, x_{n+1})}{1 - x_{n+1}}.$$

Let $\bar{\sigma}(x) = \sigma(-x)$ for $x \in S^n \backslash \{S\}$.

(a) Show that σ is bijective, and $\sigma^{-1}(u_1, \cdots, u_n) = \frac{(2u_1, \cdots, 2u_n, |u|^2 - 1)}{|u|^2 + 1}$.

(b) Compute the transition maps $\bar{\sigma} \circ \sigma^{-1}$ and $\sigma \circ \bar{\sigma}^{-1}$. Also verify that the atlas consisting of the two charts $(S^n \backslash \{N\}, \sigma)$ and $(S^n \backslash \{S\}, \bar{\sigma}$ defines a smooth structure on S^n. (The coordinates defined by σ or $\bar{\sigma}$ are called stereographic coordinates.)

(c) Show that this smooth structure is the same as the one defined in Example 1.11 (the smooth structure of S^n defined in class).

2. Find two coordinates charts on S^2 which are not smoothly compatible.

3. Compute the coordinate representation for each of the following maps, using stereographic coordinates for spheres (see Problem 1); use this to conclude that each map is smooth.

(a) $A : S^n \mapsto S^n$ is the antipodal map $A(x) = -x$

(b) $F : S^3 \mapsto S^2$ $F(z, w) = (z\overline{w} + w\overline{z}, iw\overline{z} - iz\overline{w}, z\overline{z} - w\overline{w})$, where we think of S^4 as the subset $\{(w, z) | w \in C, z \in C \text{ and } |w|^2 + |z|^2 = 1\}$.

4. Let A_1 and A_2 be the atlases for R defined by $A_1 = \{(R, Id)\}$, and $A_2 = \{(R, \psi)\}$, where $\psi(x) = x^3$. Let $f : R \mapsto R$ be any function. Determine necessary and sufficient conditions on f so that it will be

(a) a smooth map $(R, A_1) \mapsto (R, A_2)$.

(b) a smooth map $(R, A_2) \mapsto (R, A_1)$.

5. For any topological manifold M, let $C(M)$ denote the vector space of continuous functions $f : M \mapsto R$. If $F : M \mapsto N$ is a continuous map, define $F^* : C(N) \mapsto C(M)$ by $F^*(f) = f \circ F$.

(a) Show that F^* is linear.

(b) If M and N are smooth manifolds, show that F is smooth if and only if $F^*(C^\infty(N)) \subset C^\infty(M)$.

(c) If $F : M \mapsto N$ is a homeomorphism between smooth manifolds, show that it is a diffeomorphism if and only if $F^* : C^\infty(N) \mapsto C^\infty(M)$ is an isomorphism. Thus in a certain sense the entire smooth structure of M is encoded in the space $C^\infty(M)$.