Take Home Final Exam
Due: Friday, December 15

1. Let $\mathbb{RP}^n := \{\text{lines through the origin in } \mathbb{R}^{n+1}\}$. For $p \neq 0$ in $\mathbb{R}^{n+1}$, let $[p]$ be the line through p and 0.

(a) Consider the map $F : \mathbb{R}^3 \to \mathbb{R}^4$ given by $F(x, y, z) := (x^2 - y^2, xy, xz, yz)$. Prove that this induces a smooth map $f : \mathbb{RP}^2 \to \mathbb{R}^4$ given by $f([p]) := F(p)$ for $p \in S^2$. (Hint: The function $\pi : S^n \to \mathbb{RP}^n$, $p \mapsto [p]$ is smooth covering map.)

(b) Prove that $f : \mathbb{RP}^2 \to \mathbb{R}^4$ is injective. Prove that $f_* : T_{[p]}\mathbb{RP}^2 \to \mathbb{R}^4$ is injective.

(c) Prove that f is a homeomorphism onto its image (f is called the *Veronese embedding* of $\mathbb{RP}^2$ in $\mathbb{R}^4$).

2. Let $M(n, R)$ be the space of $n \times n$ matrices (with real entries). Let $F : M(n) \mapsto R$ be defined by $F(A) = det(A)$

(a) Recall that the classical adjoint of $A \in M(n, R)$ is defined by $(adjA)_{ij} = (-1)^{i+j} detA(j|i)$ where $A(j|i) \in M(n-1, R)$ is obtained by removing the $j - th$ row and $i - th$ column from A. Prove that the pushforward of F at $A \in M(n, R)$ is given by
$(F_*)_A(B) = Tr((adjA)B)$. (Hint: Compute $\frac{d}{dt}|_{t=0} f(A + tB)$).

(b) Show that 0 is the only critical value of $F : M(n, R) \mapsto R$. (Hint: $A(adjA) = (detA)I_n$, where $I_n \in M(n, R)$ is the identity matrix.)

(c) Let $SL(n, R) = \{A \in M(n, R) | detA = 1\} = F^{-1}(1)$. By (b), $SL(n, R)$ is an $n^2 - 1$ dimensional submanifold of $M(n, R)$. Describe $T_{I_n}(SL(n, R))$ (the tangent space of $SL(n, R)$ at the identify matrix I_n) explicitly as a linear subspace of $M(n, R)$.

3. Let $O(n) = \{A \in M(n, R) | A^t A = I_n\}$ We have seen that $O(n)$ is an $\frac{n(n-1)}{2}$ dimensional submanifold of $M(n, R)$. Show that

$$T_{I_n}(O(n)) = \{A \in M(n, R) | A + A^t = 0\}.$$

4. The Heisenberg group, named after Werner Heisenberg, is a group of 3×3 upper triangular matrices of the form
$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Prove that the Heisenberg group is a Lie group.

(b) Find a basis for left-invariant vector fields on the Heisenberg group.

(c) Suppose $\{X_1, X_2, X_3\}$ is the basis of left-invariant vector fields you find in (b). Compute $[X_1, X_2]$, $[X_1, X_3]$ and $[X_2, X_3]$.

(d) Find a basis for right invariant vector fields the Heisenberg group.

5. Let $\mathfrak{g}$ be a Lie algebra. A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ is called an ideal in $\mathfrak{g}$ if every $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$, $[X, Y] \in \mathfrak{h}$.

(a) If $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, show that the quotient space $\mathfrak{g}/\mathfrak{h}$ has a unique Lie algebra structure such that the projection $\pi : \mathfrak{g} \mapsto \mathfrak{g}/\mathfrak{h}$ ia a Lie algebra homomorphism.

(b) Show that a subspace $\mathfrak{h} \subset \mathfrak{g}$ is an ideal if and only if it is the kernel of a Lie algebra homomorphism.

6. Show that a connected Lie group G is Abelian if and only if the Lie bracket on its Lie algebra $\mathfrak{g}$ is zero.