HW 7 Solution

(1)

$$\text{(7.4 Problem 22)} \int \frac{3x^2+4x+3}{(x^2+1)^2}dx$$

Solution: From partial fraction $\frac{3x^2+4x+3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$, we have $3x^2+4x+3 = (Ax+B)(x^2+1)+(CX+D)$, $3x^2+4x+3 = Ax^3+AX+Bx^2+B+CX+D$ and $3x^2+4x+3 = Ax^3+Bx^2+(A+C)X+B+D$ which gives $A=0$, $B=3$, $A+C=4$ and $B+D=3$. Thus we get $A=0$ $B=3$, $C=4$ and $D=0$ and $\frac{3x^2+4x+3}{(x^2+1)^2} = \frac{3}{x^2+1}+\frac{4x}{(x^2+1)^2}$. Thus $\int \frac{3x^2+4x+3}{(x^2+1)^2}dx = \int \frac{3}{x^2+1}dx + \int \frac{4x}{(x^2+1)^2}dx = 3\arctan(x) - \frac{2}{x^2+1}+C$. We use the substitution $u=x^2+1$ and $du=2xdx$ to find $\int \frac{4x}{(x^2+1)^2}dx = \int \frac{2u}{u^2}dx = -\frac{2}{u}+C = -\frac{2}{x^2+1}+C$.

(2)

$$\int \frac{4x+3}{x^2+2x+10}dx$$

Solution: Completing the square, we get $x^2+2x+10 = (x+1)^2+3^3$. Let $x+1=3u$. Then $dx=3du$ and $x=3u-1$. $\int \frac{4x+3}{x^2+2x+10}dx = \int \frac{4x+3}{(x+1)^2+3^3}dx = \int \frac{4\cdot(3u-1)+3}{3^2u+3^3}3du = \int \frac{12u-1}{9(u^2+1)}3du = \int \frac{12u-1}{3(u^2+1)}du = \int \frac{4u}{u^2+1}du - \frac{1}{3}\int \frac{1}{u^2+1}du = 2\ln|u^2+1| - \frac{1}{3}\arctan(u)+C = 2\ln|(\frac{x+1}{3})^2+1| - \frac{1}{3}arctan(\frac{x+1}{3})+C$.

(3)

$$\int \frac{x^3-2x^2-2x+3}{x^2(x^2+1)}dx$$

Solution: From the partial fraction $\frac{x^3-2x^2-2x+3}{x^2(x^2+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1}$, we get $x^3-2x^2-2x+3 = Ax(x^2+1)+B(x^2+1)+(Cx+D)x^2$, $x^3-2x^2-2x+3 = Ax^3+Ax+Bx^2+B+Cx^3+Dx^2$ and $x^3-2x^2-2x+3 = (A+C)x^3+(B+D)x^2+Ax+B$. Thus $A+C=1$, $B+D=-2$, $A=-2$ and $B=3$ Hence$A=-2$, $B=3$, $C=3$ and $D=-5$. Thus $\frac{x^3-2x^2-2x+3}{x^2(x^2+1)} = -\frac{2}{x}+\frac{3}{x^2}+\frac{3x-5}{x^2+1}$ and $\int \frac{x^3-2x^2-2x+3}{x^2(x^2+1)}dx = -2\ln|x| - \frac{3}{x}+\frac{3}{2}\ln|x^2+1| - 5\arctan(x)+C$

(4) (Problems from Sec 7.4) Determine whether each integral is convergent or divergent. If the integral is convergent, compute its value.

(a) $\int_1^\infty \frac{1}{x^{\frac{2}{3}}}dx$

Solution: $\int_1^\infty \frac{1}{x^{\frac{2}{3}}}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^{\frac{2}{3}}}dx = \lim_{b\to\infty} 3x^{\frac{1}{3}}\Big|_1^b == \lim_{b\to\infty} 3b^{\frac{1}{3}} - 3 = \infty$. So $\int_1^\infty \frac{1}{x^{\frac{2}{3}}}dx$ diverges.

(b) $\int_e^\infty \frac{1}{x\ln x}dx$

Solution: $\int_e^\infty \frac{1}{x\ln x}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x\ln x}dx = \lim_{b\to\infty}\ln|\ln x|\Big|_e^b == \lim_{b\to\infty}\ln|\ln b| - \ln|\ln e| = \infty$. So $\int_e^\infty \frac{1}{x\ln x}dx$ diverges. We have used $u=\ln x$ and $du=\frac{1}{x}dx$ to integrate $\int \frac{1}{x\ln x}dx = \int \frac{1}{u}du = \ln|u|+C = \ln|\ln x|+C$.

- **(c)** $\int_e^\infty \frac{1}{x(\ln x)^2}dx$ Solution: We have used $u = \ln x$ and $du = \frac{1}{x}dx$ to integrate $\int \frac{1}{x(\ln x)^2}dx = \int \frac{1}{u^2}du = -\frac{1}{u} + C = -\frac{1}{\ln x} + C$ and $\int_e^\infty \frac{1}{x(\ln x)^2}dx = \lim_{b\to\infty} -\frac{1}{\ln x}\Big|_e^b = \lim_{b\to\infty} -\frac{1}{\ln b} + 1 = 0 + 1 = 1$.
- **(d)** $\int_e^\infty \frac{\ln x}{x^2}dx$ Integration by parts $u = \ln x$, $dv = \frac{1}{x^2}dx$, $du = \frac{1}{x}dx$ and $v = \int \frac{1}{x^2}dx = -\frac{1}{x} + C$, we get $\int \frac{\ln x}{x^2}dx = -\frac{\ln x}{x} - \int(-\frac{1}{x}) \cdot \frac{1}{x}dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}dx = -\frac{\ln x}{x} - \frac{1}{x} + C$. So $\int_e^\infty \frac{\ln x}{x^2}dx = \lim_{b\to\infty} -\frac{\ln x}{x} - \frac{1}{x}\Big|_e^b \lim_{b\to\infty} -\frac{\ln b}{b} - \frac{1}{b} - (-\frac{\ln e}{e} - \frac{1}{e}) = \frac{2}{e}$. We have used $\ln e = 1$, $\lim_{b\to\infty} \frac{1}{b} = 0$ and $\lim_{b\to\infty} \frac{\ln b}{b} = \lim_{b\to\infty} \frac{(\ln b)'}{b'} = \lim_{b\to\infty} \frac{\frac{1}{b}}{1} = \lim_{b\to\infty} \frac{1}{b} = 0$
- **(e)** $\int_{-\infty}^\infty x^5 dx$ Since $\int_0^\infty x^5 dx = \lim_{b\to\infty} \int_0^b x^5 dx \lim_{b\to\infty} \frac{b^6}{6} = \infty$, so $\int_{-\infty}^\infty x^5 dx$ diverges.