# HW 8 Solution

(1) (8.1 Problem 30) Solve the differential equation $\frac{dy}{dx} = \frac{1}{2}y^2 - 2y$ with $y(0) = -3$. Note that $\frac{1}{2}y^2 - 2y = \frac{1}{2}y(y-4)$.
Solution: Note that $\frac{dy}{dx} = \frac{1}{2}y^2 - 2y = \frac{1}{2}y(y-4)$. Separating the variable, we have $\int \frac{1}{y(y-4)}dy = \int \frac{1}{2}dx$. $\frac{1}{y(y-4)} = \frac{A}{y} + \frac{B}{y-4}$. Multiplying $y(y-4)$ to both sides, we have $1 = A(y-4) + By$. Plugging $y = 0$ and $y = 4$, we have $A = -\frac{1}{4}$ and $B = \frac{1}{4}$. Thus $\frac{1}{y(y-4)} = -\frac{1}{4} \cdot \frac{1}{y} + \frac{1}{4} \cdot \frac{1}{y-4}$ and $\int \frac{1}{y(y-4)}dy = \int (-\frac{1}{4} \cdot \frac{1}{y} + \frac{1}{4} \cdot \frac{1}{y-4})dy = -\frac{1}{4}\ln|y| + \frac{1}{4}\ln|y-4| = \frac{1}{4}\ln|\frac{y-4}{y}| + c$. Thus $\int \frac{1}{y(y-4)}dy = \int \frac{1}{2}dx$ can be integrated to get $\frac{1}{4}\ln|\frac{y-4}{y}| = \frac{x}{2} + c$, $\ln|\frac{y-4}{y}| = 2x + C$ (here $C = 4c$) $\frac{y-4}{y} = e^{2x+C} = e^C \cdot e^{2x} = De^{2x}$ where $D = e^C$. From $\frac{y-4}{y} = De^{2x}$, we have $y - 4 = De^{2x}y$, $y - De^{2x}y = 4$, $y(1 - De^{2x}) = 4$ and $y = \frac{4}{1-De^{2x}}$. Using the initial condition $y(0) = -3$, we have $\frac{4}{1-De^{2\cdot 0}} = -3$, $\frac{4}{1-D} = -3$, $4 = -3 + 3D$, $3D = 7$ and $D = \frac{7}{3}$. Thus $y = \frac{4}{1-\frac{7}{3}e^{2x}}$.

(2) (8.1 Problem 48) Solve the differential equation $\frac{dy}{dx} = x^2y^2$ with $y(1) = 1$
Solution: Separating the variable, we have $\int \frac{1}{y^2}dy = \int x^2 dx$, $-\frac{1}{y} = \frac{x^3}{3} + c$ and $y = -\frac{1}{\frac{x^3}{3}+c}$. Using the initial condition $y(1) = 1$, we have $-\frac{1}{\frac{1}{3}+c} = 1$, $\frac{1}{3} + c = -1$ and $c = -\frac{4}{3}$. Thus $y = -\frac{1}{\frac{x^3}{3} - \frac{4}{3}} = -\frac{1}{\frac{x^3-4}{3}} = -\frac{3}{x^3-4}$.

(3) (Part of 8.1 Problem 40) Suppose that the size of a population, denoted by $N(t)$, satisfies

$$\frac{dN}{dt} = 0.7N(1 - \frac{N}{35}). \tag{0.0.1}$$

**(a)** Determine all equilibria by solving $\frac{dN}{dt} = 0$. Solution: Solving $0.7N(1 - \frac{N}{35}) = 0$, we have $N = 0$ and $N = 35$.

**(b)** Solve the differential equation (0.0.1) with $N(0) = 10$ and find $\lim_{t\to\infty} N(t)$. Solution: We can rewrite $0.7N(1 - \frac{N}{35}) = 0.7N(\frac{35-N}{35}) = -\frac{0.7}{35}N(N-35) = -0.02N(N-35)$. So the diff eq $\frac{dN}{dt} = 0.7N(1 - \frac{N}{35})$ is the same as $\frac{dN}{dt} = -0.02N(N-35)$. Separating the variable, we have $\int \frac{1}{N(N-35)}dN = \int -0.02dt$. Let $\frac{1}{N(N-35)} = \frac{A}{N} + \frac{B}{N-35}$. Multiplying $N(N-35)$ to both sides, we have $1 = A(N-35) + BN$. Plugging $N = 0$ and $N = 35$, we have $A = -\frac{1}{35}$ and $B = \frac{1}{35}$. So $\frac{1}{N(N-35)} = -\frac{1}{35} \cdot \frac{1}{N} + \frac{1}{35} \cdot \frac{1}{N-35}$ and $\int \frac{1}{N(N-35)}dN = -\frac{1}{35}\int \frac{1}{N}dN + \frac{1}{35}\int \frac{1}{N-35}dN = -\frac{1}{35}\ln|N| + \frac{1}{35}\ln|N-35| = \frac{1}{35}\ln|\frac{N-35}{N}| + c$. Thus $\int \frac{1}{N(N-35)}dN = \int -0.02dt$ can be integrated as $\frac{1}{35}\ln|\frac{N-35}{N}| = -0.02t + c$ and $\ln|\frac{N-35}{N}| = -0.7t + c_1$, $\frac{N-35}{N}Ce^{-0.7t}$, $N - 35 = Ce^{-0.7t}N$, $N - Ce^{-0.7t}N = 35$, $N(1 - Ce^{-0.7t}) =$

35 and $N = \frac{35}{1-Ce^{-0.7t}}$. Using the condition $N(0) = 10$, we have $\frac{35}{1-Ce^{0}} = 10$, $\frac{35}{1-C} = 10$, $35 = 10 - 10C$, $10C = -25$ and $C = -2.5$. Thus $N = \frac{35}{1-(-2.5)e^{-0.7t}} = \frac{35}{1+2.5e^{-0.7t}}$ and $\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \frac{35}{1+2.5e^{-0.7t}} = 35$

**(c)** Solve the differential equation (0.0.1) with $N(0) = 50$ and find $\lim_{t\to\infty} N(t)$.
Solution: Using the general solution $N = \frac{35}{1-Ce^{-0.7t}}$ and $N(0) = 50$, we have $\frac{35}{1-C} = 50$, $35 = 50 - 50C$, $50C = 15$ and $C = 0.3$. Thus $N = \frac{35}{1-0.3e^{-0.7t}}$ and $\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \frac{35}{1-0.3e^{-0.7t}} = 35$.

(4) (Part of 8.2 Problem 4) Suppose that $\frac{dy}{dx} = y(2-y)(y-3)$

**(a)** Determine the equilibria of this differential equation. Solution: Solving $y(2-y)(y-3) = 0$, we have $y = 0$, $y = 2$ or $y = 3$. So the equilibria of this differential equation are $y = 0$, $y = 2$ or $y = 3$

**(b)** Graph $\frac{dy}{dx}$ as a function of $y$, and use your graph to discuss the stability of the equilibria.
Solution: We plug in the value $y = -1 < 0$, $0 < y = 1 < 2$ and $2 < y = 2.5 < 3$ and $3 < y = 4$ to $y(2-y)(y-3)$.

| y | -1 | 1 | 2.5 | 4 |
|---|---|---|---|---|
| y(2-y)(y-3) | (-1)(2-(-1))(-1-3) | 1(2-1)(1-3) | 2.5(2-2.5)(2.5-3) | 4(2-4)(4-3) |
| | + | - | + | - |

From the graph on next page, we know that $y = 0$ is stable, $y = 2$ is unstable, $y = 3$ is stable.

(5) (sec 8.2 ) Suppose that $\frac{dy}{dx} = g(y)$ and the graph of $\frac{dy}{dx}$ as a function of $y$ is given by the figure above

**(a)** Determine the equilibria of this differential equation. Solution: The equilibrium of this differential equation are $y = 2$, $y = 4$, $y = 7$ and $y = 9$

**(b)** Use the graph to discuss the stability of the equilibria. From the graph on next page, we know that $y = 2$ is unstable, $y = 4$ is unstable, $y = 7$ is stable and $y = 9$ is unstable.

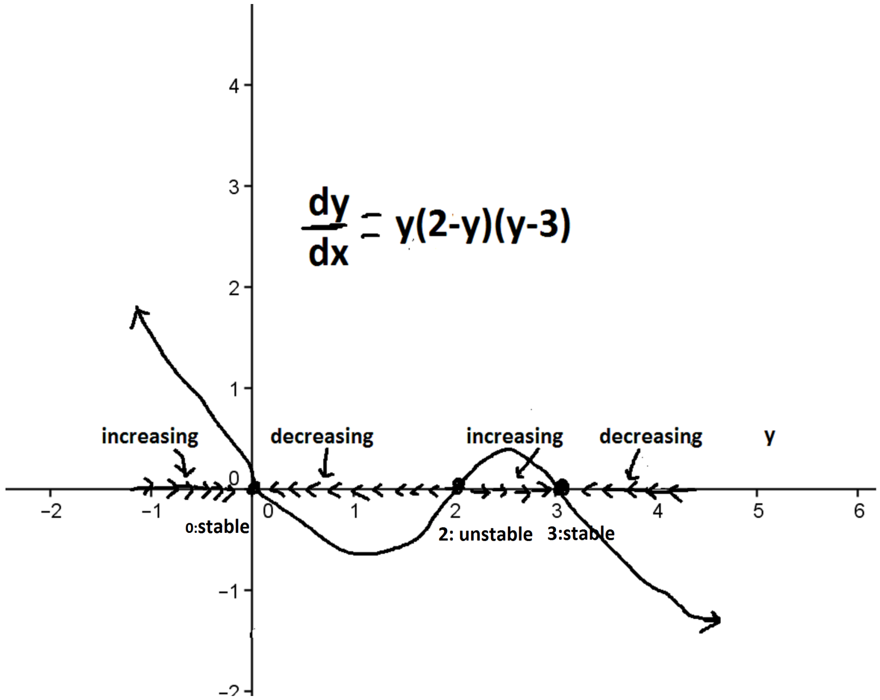


FIGURE 1. graph for Problem 4

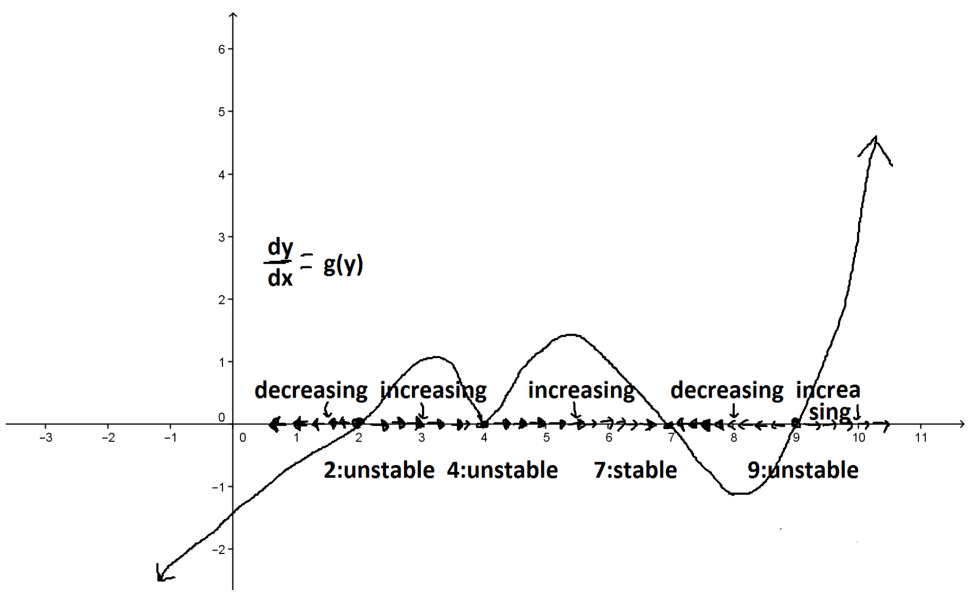


FIGURE 2. graph for Problem 5